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An Extension of VIKOR Approach for MCDM Using Bipolar Fuzzy Preference δ-Covering Based Bipolar Fuzzy Rough Set Model

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ABSTRACT

Bipolarity highlights the positive and negative facets of a certain dilemma. This script aims to propose a novel multi-criteria decision-making (MCDM) approach based on bipolar fuzzy preference δ-covering based bipolar fuzzy rough set (BFPδC-BFRS) model by combining the VIKOR (VIseKriterijumska Optimizacija I Kompromisno Rasenje) scheme. The VIKOR scheme is viewed as a beneficial MCDM strategy, particularly in situations where an expert is incapable of making the right decision at the beginning of system design. The VIKOR method works well for problems with competing attributes because it operates under the presumptions that compromise is acceptable in conflict resolution, the expert seeks a solution that is extremely close to the best, and all developed attributes are taken into consideration when processing the various alternatives. In this study, firstly, we proposed an integrated MCDM based on BFPδC-BFRSs using the VIKOR methodology. Moreover, we solve a real-world illustration to show the effectiveness of the expanded VIKOR approach. Finally, we demonstrate a detailed comparative analysis of the proposed methodology with some prevalent decision-making approaches to substantiate the accountability of the recommended scheme.

1. Introduction

The actual world is full of indeterminacy and uncertainty. Rather than dealing with specific problems, we typically deal with ambiguous ones. The success of traditional techniques is not always guaranteed because of the inherent uncertainties in these issues. Zadeh [1] promulgated the paradigm of the fuzzy set (FS), which unlocked the avenues for scholars to combat the uncertainty of data. FS theory relies on membership function (MF), which permits us to assess the membership grade (MG) of an item regarding a set. The bigger the MG, the higher the association of that item to the related set. Many remarkable applications of FSs can be seen in [2–5].

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It has been acknowledged that RS theory [6, 7] is a useful mathematical approach to managing intelligent systems that display ambiguity and uncertainty. The effectiveness of this comparatively emerging soft computing tool has been effectively demonstrated and leveraged in a variety of fields, including pattern recognition, conflict resolution, knowledge discovery, data mining, image processing, medical science, neural networks, and so forth. It has garnered a lot of attention in recent years. The indiscernibility relation between arbitrary items is described by equivalence relation (ER), the central and fundamental idea of the RS theory. Although RS theory has been effectively utilized in many areas, certain issues might lead the theory's application scope to be limited. These flaws could be the consequence of inaccurate knowledge of the items being evaluated. It might be difficult to find an ER with a piece of incomplete information. Consequently, RS models have developed several fascinating generalizations under various circumstances in recent years, such as an RS based on tolerance relations [8], RS based on neighbourhood operators [9], fuzzy rough set (FRS) [10], rough fuzzy set (RFS) model [11], dominance-based RSs [12], variable precision RS [13], covering-based RSs [14] and grey tolerance RS [15].

It is critical to keep in mind that everything has two aspects and that fuzziness and bipolarity are innate characteristics of human perception. Cognitive psychology research indicates that bipolar reasoning is essential to human cognition actions. The areas of the brain responsible for positive and negative impacts do not appear to be situated there. Experts in a variety of fields have noted the importance of bipolarity, including database querying, decision-making, and categorization. Fuzziness and bipolarity are two separate but correlative concepts intended to mimic certain facets of human cognition. The latter emphasizes the polarity and importance of the facts, whereas the former concentrates on linguistic imprecision. Recent research has consistently demonstrated the great importance of two concepts. In light of this, Zhang [16] postulated bipolar FSs (BFSs) as an extension of FSs with an MG lies in $[-1, 1]$. An object's MG of "0" indicates that it does not pertain to the associated property; an object's MG of $(0, 1]$ means that it partially fulfils the property; and an object's MG of $[-1, 0)$ signifies that it partially satisfies the implicit counter-property. BFS theory has been widely applied to solve practical issues. Additionally, several initiatives have been undertaken to integrate RS and BFS theories. Han *et al.* [17] devised a bipolar-valued RFS version with application in decision analysis. Yang *et al.* [18] recommended a bipolar FRS model on dual universes. An MCDM technique employing interval-valued bipolar fuzzy information was first introduced by Wei *et al.* [19]. Ali *et al.* [20] studied attribute reductions of bipolar fuzzy relation (BFR). By using the (α, β)-indiscernibility of a BFR, Gul and Shabir [21] proposed a unique method for the roughness of a crisp set. A bipolar FRS model with inconsistent bipolarity in two universes was proposed by Han *et al.* [22]. Jana and Pal [23] studied a robust bipolar fuzzy EDAS scheme. Malik and Shabir [24] launched a consensus approach via rough BFSs. Luo and Hu [25] integrated a bipolar three-way decision scheme with applicability in analyzing incomplete information. Recently, Gul *et al.*[26] devised a bipolar fuzzy preference δ-covering based bipolar FRS (BFPδC-BFRS) model and corresponding decision-making applications.

In light of the literature mentioned above, we observed that a plethora of researchers put forward several hybrid models of RSs, FS, and BFSs. However, to the best of our knowledge, no prior investigation has been conducted into the hybridization of VIKOR methodology and BFPδC-BFRS model. To bridge this knowledge gap, in this study, we studied an extension of the VIKOR scheme for MCDM under the BFPδC-BFRS model.

The rest of this study is systematized as follows: In Section 2, we revisit some rudimentary ideas linked to FSs, BFSs, and RSs. In Section 3, we first proposed an integrated MCDM scheme based on the BFPδC-BFRS model and the VIKOR approach. Besides, an algorithm for MCDM is also elaborated. In Section 4, an applied example of decision-making using bipolar fuzzy information is provided. In Section 5, we perform a comparative study of our projected methodology with other prevailing schemes. Section 6 provides conclusions.

2. Preliminaries

In this segment, we review numerous essential terminologies which will be utilized in our study.

Definition 2.1 *[6] An ordered pair (Υ, σ) is named as an approximation space, where Υ is a non-void finite set and σ is an ER over Υ. For $\mathcal{H} \subseteq \Upsilon$, $\mathcal{H}$ may or may not be stated as a union of some equivalence classes of σ. If it is possible to express $\mathcal{H}$ as the latter, then it is named definable; if not, it is an RS. If $\mathcal{H}$ is an RS, then it can be approximated as:*

$$\left.\begin{array}{l}\sigma_*(\mathcal{H}) = \{\varepsilon \in \Upsilon : [\varepsilon]_\sigma \subseteq \mathcal{H}\}, \\ \sigma^*(\mathcal{H}) = \{\varepsilon \in \Upsilon : [\varepsilon]_\sigma \cap \mathcal{H} \neq \varnothing\},\end{array}\right\} \tag{1}$$

which are referred to as lower and upper approximations of $\mathcal{H}$, respectively, where

$$[\varepsilon]_\sigma = \{\varepsilon' \in \Upsilon : (\varepsilon, \varepsilon') \in \sigma\}. \tag{2}$$

Further, the set

$$\mathcal{B}nd_\sigma(\mathcal{H}) = \sigma^*(\mathcal{H}) - \sigma_*(\mathcal{H}) \tag{3}$$

is called the boundary region of $\mathcal{H}$ in Υ. Obviously, $\mathcal{H}$ is definable if $\sigma_*(\mathcal{H}) = \sigma^*(\mathcal{H})$. $\mathcal{H}$ is a RS if $\sigma_*(\mathcal{H}) \neq \sigma^*(\mathcal{H})$.

Definition 2.2 *[1] A FS F on Υ is a function $F : \mho \longrightarrow [0, 1]$. The value $F(\varepsilon)$ of F for $\varepsilon \in \Upsilon$ signifies the MG of ε in F.*

Definition 2.3 *[16] A BFS Ψ over Υ is postulated as:*

$$\Psi = \left\{\langle \varepsilon, \Psi^+(\varepsilon), \Psi^-(\varepsilon)\rangle : \varepsilon \in \Upsilon\right\}, \tag{4}$$

where $\Psi^+ : \Upsilon \longrightarrow [0, 1]$ and $\Psi^- : \Upsilon \longrightarrow [-1, 0]$ are said to be positive MG (PMG) and negative MG (NMG), respectively. The PMG reveals the satisfaction degree of an item ε to the property, and the NMG demonstrates the satisfaction degree of ε to some implicit counter-property related to a BFS Ψ.

From now on, the assembling of all BFSs over Υ is denoted by $\mathcal{BF}(\Upsilon)$.

Definition 2.4 *[16] Let $\Psi, \Gamma \in \mathcal{BF}(\Upsilon)$. Then $\forall \varepsilon \in \Upsilon$,*

1. $\Psi \subseteq \Gamma \Leftrightarrow \Psi^+(\varepsilon) \leq \Gamma^+(\varepsilon)$ *and* $\Psi^-(\varepsilon) \geq \Gamma^-(\varepsilon)$;
2. $\Psi = \Gamma \Leftrightarrow \Psi^+(\varepsilon) = \Gamma^+(\varepsilon)$ *and* $\Psi^-(\varepsilon) = \Gamma^-(\varepsilon)$;
3. $(\Psi \cap \Gamma)(\varepsilon) = \{\Psi^+(\varepsilon) \wedge \Gamma^+(\varepsilon), \Psi^-(\varepsilon) \vee \Gamma^-(\varepsilon)\}$;
4. $(\Psi \cup \Gamma)(\varepsilon) = \{\Psi^+(\varepsilon) \vee \Gamma^+(\varepsilon), \Psi^-(\varepsilon) \wedge \Gamma^-(\varepsilon)\}$;
5. $\Psi^c(\varepsilon) = \{1 - \Psi^+(\varepsilon), -1 - \Psi^-(\varepsilon)\}$.

Definition 2.5 *[16] A BFR $\mathbb{B}$ over Υ is a BFS over $\Upsilon \times \Upsilon$. Thus, it may be formulated as:*

$$\mathbb{B} = \left\{\langle (x', y'), \mu^+_{\mathbb{B}}(x', y'), \mu^-_{\mathbb{B}}(x', y')\rangle : (x', y') \in \Upsilon \times \Upsilon\right\}, \tag{5}$$

where $\mu^+_{\mathbb{B}} : \Upsilon \times \Upsilon \longrightarrow [0, 1]$ and $\mu^-_{\mathbb{B}} : \Upsilon \times \Upsilon \longrightarrow [-1, 0]$.

Recently, Gul *et al*. [26] generated the paradigm of bipolar fuzzy preference relation (BFPR) along with its core features.

Definition 2.6 *[26] A BFPR $\mathfrak{B}$ is a BFS over $\Upsilon \times \Upsilon$, which is expressed by its positive and negative MFs given as $\mu^{+}_{\mathfrak{B}} : \Upsilon \times \Upsilon \longrightarrow [0,1]$ and $\mu^{-}_{\mathfrak{B}} : \Upsilon \times \Upsilon \longrightarrow [-1,0]$. For $\Upsilon = \{\flat_1, \flat_2, \cdots, \flat_n\}$, we can denote it as:*

$$\mathfrak{B} = \Big[\langle a_{ij}, b_{ij}\rangle\Big]_{n\times n} = \begin{array}{c} \\ \flat_1 \\ \flat_2 \\ \vdots \\ \flat_n \end{array} \begin{pmatrix} \flat_1 & \flat_2 & \cdots & \flat_n \\ \langle a_{11}, b_{11}\rangle & \langle a_{12}, b_{12}\rangle & \cdots & \langle a_{1n}, b_{1n}\rangle \\ \langle a_{21}, b_{21}\rangle & \langle a_{22}, b_{22}\rangle & \cdots & \langle a_{2n}, b_{2n}\rangle \\ \vdots & \ddots & \vdots & \\ \langle a_{n1}, b_{n1}\rangle & \langle a_{n2}, b_{n2}\rangle & \cdots & \langle a_{nn}, b_{nn}\rangle \end{pmatrix},$$

where $\langle a_{ij}, b_{ij}\rangle$ indicates the bipolar fuzzy preference degree (BFPD) of x_i over x_j, $a_{ij} \in [0,1]$, $b_{ij} \in [-1,0]$. Further, a_{ij} and b_{ij} fulfills the constraints, $a_{ij} + a_{ji} = 1, b_{ij} + b_{ji} = -1, a_{ii} = 0.5$ and $b_{ii} = -0.5\ \forall i,j = 1,2,\cdots,n$. Specially,

- *$a_{ij} = 0.5, b_{ij} = -0.5$ indicates indifference between $\flat_i$ and $\flat_j$;*
- *$a_{ij} > 0.5, b_{ij} > -0.5$ reveals that $\flat_i$ is better than $\flat_j$;*
- *$a_{ij} < 0.5, b_{ij} < -0.5$ signifies that $\flat_j$ is better than $\flat_i$;*
- *$a_{ij} = 1, b_{ij} = 0$ suggests that $\flat_i$ is absolutely better than $\flat_j$;*
- *$a_{ij} = 0, b_{ij} = -1$ means $\flat_j$ is absolutely better than $\flat_i$.*

Definition 2.7 *[26] A BFPR $\mathfrak{B} = \Big[\langle a_{ij}, b_{ij}\rangle\Big]_{n\times n}$ is termed as an additive consistent, if $\forall i,j,k \in \{1,2,\cdots,n\}$ the following conditions hold:*

1. *$a_{ij} = a_{ik} - a_{jk} + 0.5$,*
2. *$b_{ij} = b_{ik} - b_{jk} + 0.5$.*

Definition 2.8 *[26] Let $\Upsilon = \{\flat_i : i = 1,2,\cdots,n\}$ be the universe of n items and $C = \{C_k : k = 1,2,\cdots,m\}$ be a finite collection of m criteria. Let $f : \Upsilon \times C \longrightarrow [0,1]$ be a fuzzy MF and $g : \Upsilon \times C \longrightarrow [-1,0]$ be a fuzzy non-membership function. Then, we postulate the transfer functions to determine the BFPD of any two items $\flat_i, \flat_j \in \Upsilon$ regarding the criterion C_k as follows:*

$$a^{C_k}_{ij} = \frac{f(\flat_i, C_k) - f(\flat_j, C_k) + 1}{2}, \tag{6}$$

$$b^{C_k}_{ij} = \frac{g(\flat_j, C_k) - g(\flat_i, C_k) - 1}{2}. \tag{7}$$

For a BFPR $\mathfrak{B}_{C_k}(\flat_i, \flat_j) = \Big[\Big\langle a^{C_k}_{ij}, b^{C_k}_{ij}\Big\rangle\Big]_{n\times n}$, the transfer functions (6) and (7) satisfied the following characteristics for all $\flat_i, \flat_j, \flat_k \in \Upsilon$:

1. $a_{ii}^{C_k} = 0.5$ *and* $b_{ii}^{C_k} = -0.5$.
2. $a_{ij}^{C_k} + a_{ji}^{C_k} = 1$ *and* $b_{ij}^{C_k} + b_{ji}^{C_k} = -1$.
3. $a_{ij}^{C_k} + a_{j\ell}^{C_k} = a_{i\ell}^{C_k} + 0.5$ *and* $b_{ij}^{C_k} + b_{j\ell}^{C_k} = b_{i\ell}^{C_k} - 0.5$.

Table 1 illustrates a bipolar fuzzy matrix, where $\Upsilon = \{\flat_i : i = 1, 2, \cdots, 5\}$ and $C = \{C_1, C_2\}$. Based on criteria C_1 and C_2, we can construct the BFPRs of alternative $\flat_i$ to the alternative $\flat_j (i, j = 1, 2, \cdots, 5)$ by employing formulas (6) and (7), we acquired:

Table 1
Bipolar fuzzy matrix

Υ/C	C_1	C_2
x_1	(0.5, - 0.25)	(0.8, - 0.7)
x_2	(0.25, - 0.8)	(0.9, - 0.4)
x_3	(0.33, - 0.25)	(0.75, - 0.4)
x_4	(0.65, - 0.6)	(0.3, - 0.75)
x_5	(1, - 0.5)	(0.4, - 0.35)

$$\mathfrak{B}_{C_1}(\flat_i, \flat_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.625, -0.775 \rangle & \langle 0.585, -0.500 \rangle & \langle 0.425, -0.675 \rangle & \langle 0.250, -0.625 \rangle \\ \langle 0.375, -0.225 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.460, -0.225 \rangle & \langle 0.300, -0.400 \rangle & \langle 0.125, -0.350 \rangle \\ \langle 0.415, -0.500 \rangle & \langle 0.540, -0.775 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.340, -0.675 \rangle & \langle 0.165, -0.625 \rangle \\ \langle 0.575, -0.325 \rangle & \langle 0.700, -0.600 \rangle & \langle 0.660, -0.325 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.325, -0.450 \rangle \\ \langle 0.750, -0.375 \rangle & \langle 0.875, -0.650 \rangle & \langle 0.835, -0.375 \rangle & \langle 0.675, -0.550 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}, \tag{8}$$

$$\mathfrak{B}_{C_2}(\flat_i, \flat_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.450, -0.350 \rangle & \langle 0.525, -0.350 \rangle & \langle 0.750, -0.525 \rangle & \langle 0.700, -0.325 \rangle \\ \langle 0.550, -0.650 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.575, -0.500 \rangle & \langle 0.800, -0.675 \rangle & \langle 0.750, -0.475 \rangle \\ \langle 0.475, -0.650 \rangle & \langle 0.425, -0.500 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.725, -0.675 \rangle & \langle 0.675, -0.475 \rangle \\ \langle 0.250, -0.475 \rangle & \langle 0.200, -0.325 \rangle & \langle 0.275, -0.325 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.450, -0.300 \rangle \\ \langle 0.300, -0.675 \rangle & \langle 0.250, -0.525 \rangle & \langle 0.325, -0.525 \rangle & \langle 0.550, -0.700 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}. \tag{9}$$

Definition 2.9 *[26] The bipolar fuzzy preference classes (BFPCs) of an item $x_i \in \Upsilon$ induced by a BFPR $\mathfrak{B}_{C_k}(x_i, x_j) = \left[\left\langle a_{ij}^{C_k}, b_{ij}^{C_k} \right\rangle\right]_{n \times n}$ is portrayed as:*

$$[x_i]_{\mathfrak{B}_{C_k}} = \frac{\left\langle a_{i1}^{C_k}, b_{i1}^{C_k} \right\rangle}{x_1} + \frac{\left\langle a_{i2}^{C_k}, b_{i2}^{C_k} \right\rangle}{x_2} + \cdots + \frac{\left\langle a_{in}^{C_k}, b_{in}^{C_k} \right\rangle}{x_n}. \tag{10}$$

The BFPR $\mathfrak{B}_{C_k}$ produces a collection of bipolar fuzzy information granules from the universe, which constitutes the bipolar fuzzy preference granular structure described as follows:

$$P\left(\mathfrak{B}_{C_k}\right) = \left\{ [x_1]_{\mathfrak{B}_{C_k}}, [x_2]_{\mathfrak{B}_{C_k}}, \cdots, [x_n]_{\mathfrak{B}_{C_k}} \right\}.$$

Gul *et al.* [26] implemented the formulas (6) and (7) to originate the conception of bipolar fuzzy preference δ-neighborhood (BFPδ-nghd) in a $BFP\delta C$-approximation space ($BFP\delta C$-AS) and examined its associated features. Eventually, they design an innovative BFPδC-BFRS variant via BFPδ-nghd.

Definition 2.10 *[26] Presume Υ be a non-void finite universe and $P(\mathfrak{B}_{C_k})$ be a bipolar fuzzy preference granular structure. Then for every $\delta = \langle \alpha, \beta \rangle \in (0,1] \times [-1,0)$, we name $P(\mathfrak{B}_{C_k})$ a $BFP\delta C$ of Υ, when*

$$\left(\bigcup_{i,j=1}^{n} a_{ij}^{C_k}\right)(x) \geq \alpha \text{ and } \left(\bigcap_{i,j=1}^{n} b_{ij}^{\mathfrak{C}_k}\right)(x) \leq \beta,\ \forall\, x \in \Upsilon. \tag{11}$$

Furthermore, the pair $(\mho, P(\mathfrak{B}_{C_k}))$ is termed a $BFP\delta C$-AS.

Definition 2.11 *[26] Let $(\mho, P(\mathfrak{B}_{C_k}))$ be a $BFP\delta C$-AS. For each $x \in \Upsilon$, we characterize the BFPδ-nghd $\aleph_x^\delta$ of x as:*

$$\aleph_x^\delta = \left\langle \aleph_x^\alpha, \aleph_x^\beta \right\rangle, \tag{12}$$

where,

$$\aleph_x^\alpha = \bigwedge \left\{ [x_i]_{\mathfrak{B}_{C_k}} : a_{ij}^{C_k} \geq \alpha \right\}, \tag{13}$$

and

$$\aleph_x^\beta = \bigvee \left\{ [x_i]_{\mathfrak{B}_{C_k}} : b_{ij}^{C_k} \leq \beta \right\}. \tag{14}$$

Definition 2.12 *[26] Assume that $(\Upsilon, P(\mathfrak{B}_{C_k}))$ be a $BFP\delta C$-AS and $\delta = \langle \alpha, \beta \rangle \in (0,1] \times [-1,0)$. The $BFP\delta C$ lower and upper approximations of a BFS $\Psi = \langle \Psi^+, \Psi^- \rangle$ in Υ regarding $(\mho, P(\mathfrak{B}_{C_k}))$ are respectively postulated as:*

$$\left.\begin{aligned} \underline{BF}_C(\Psi) &= \left\langle \underline{(\Psi^+)(x)}_C, \underline{(\Psi^-)(x)}_C \right\rangle, \\ \overline{BF}_C(\Psi) &= \left\langle \overline{(\Psi^+)(x)}_C, \overline{(\Psi^-)(x)}_C \right\rangle, \end{aligned}\right\} \tag{15}$$

where,

$$\left.\begin{aligned} \underline{(\Psi^+)(x)}_C &= \bigwedge_{y\in\Upsilon} \left\{ \left(1 - {}^{C}\aleph_x^\alpha(y)\right) \vee \Psi^+(y) \right\}, \\ \underline{(\Psi^-)(x)}_C &= \bigvee_{y\in\Upsilon} \left\{ {}^{C}\aleph_x^\beta(y) \wedge \Psi^-(y) \right\}, \\ \overline{(\Psi^+)(x)}_C &= \bigvee_{y\in\Upsilon} \left\{ {}^{C}\aleph_x^\alpha(y) \wedge \Psi^+(y) \right\}, \\ \overline{(\Psi^-)(x)}_C &= \bigwedge_{y\in\Upsilon} \left\{ \left(-1 - {}^{\mathfrak{C}}\aleph_x^\beta(y)\right) \vee \Psi^-(y) \right\}, \text{for every } x \in \Upsilon. \end{aligned}\right\} \tag{16}$$

If $\underline{BF}_C(\Psi) \neq \overline{BF}_C(\Psi)$, then Ψ is named BFPδC-BFRS; else, Ψ is termed a bipolar fuzzy definable.

3. An Integrated MCDM Using BFPδC-BFRS Model and VIKOR

In the current segment, we introduced a novel MCDM strategy by unifying the BFPδC- BFRS model and the VIKOR scheme.

Consider a collection $\Upsilon = \{x_i : i = 1, 2, \cdots, n\}$ of n items and $C = \{C_k : k = 1, 2, \cdots, m\}$ be a collection of m criteria that are decided by a decision expert. The associated weight vector of all criteria is signified by $\mathfrak{W} = (\hbar_1, \hbar_2, \cdots, \hbar_m)^T$ such that $0 \le \hbar_j \le 1$ and $\sum_{j=1}^{m} \hbar_j = 1$. Let the BFS $\Psi = \{\langle x, \Psi^+(x), \Psi^-(x)\rangle : x \in \Upsilon\}$ over Υ be the description of all items by the expert. Let $\mathbb{E}$ signify a finite set of the domain for the information functions $f(x_\ell, C_k)$ and $g(x_\ell, C_k)$, where $f(x_\ell, C_k) \in [0, 1]$ stands for the PMG of x_ℓ w.r.t. C_k given by the expert and $g(x_\ell, C_k) \in [-1, 0]$ stands for the NMG of x_ℓ w.r.t. C_k given by the expert. The BFPC $[x_i]_{\mathfrak{B}_{C_k}}(x_j)$ stands for the efficacy value of x_j's w.r.t. C_k. For a critical value $\delta = \langle \alpha, \beta\rangle \in (0, 1] \times [-1, 0)$, let for each $x_i \in \Upsilon$, there is at least one criteria $C_k \in C$ such that the efficacy value of x_j for the criteria C_k is not less than α and greater than β, and $P(\mathfrak{B}_{C_k})$ is a $BFP\delta C$ of Υ. Then the BFPδ-nghd $\aleph^{\delta}_{x_j}$ of x_j regarding C_k is a BFS described as:

$$^{C_k}\aleph^{\delta}_{x_j}(x_r) = \left[\bigwedge\left\{a^{C_k}_{ij} : a^{C_k}_{ij} \ge \alpha\right\}, \bigvee\left\{b^{C_k}_{ij} : b^{C_k}_{ij} \le \beta\right\}\right](x_r); r = 1, 2, \cdots, n, \tag{17}$$

which yields the minimum of all efficacy values for each x_r w.r.t. C_k.

If $\Psi = \{\langle x_i, \Psi^+(x_i), \Psi^-(x_i)\rangle : x_i \in \Upsilon\} \in \mathcal{BF}(\Upsilon)$ represent PMG and NMG of each $x_i \in \Upsilon$ given by the expert. The decision-making for the MCDM problem is how to select an optimal alternative among all. We denote the bipolar fuzzy information system $(\Upsilon, C, \mathfrak{W}, \mathbb{E})$.

3.1 Decision-making Methodology

The recommended method has subsequent steps.

Step 1: Assume that the decision-maker furnish his/her assessment of the alternatives x_i regarding each criterion C_k by considering bipolar fuzzy numbers as $x_{ik} = \langle \Psi^+_k(x_i), \Psi^-_k(x_i)\rangle$, where the PMG $\Psi^+_k(x_i)$ signifies the satisfaction degree of alternatives x_i to the criteria C_k and the NMG $\Psi^-_k(x_i)$ denotes the satisfaction degree of the alternatives x_i to some counter property of the criteria C_k. Formally, a bipolar fuzzy MCDM problem can be represented by an $n \times m$ matrix as follows:

$$\mathfrak{D} = \left[\left\langle \Psi^+_k(x_i), \Psi^-_k(x_i)\right\rangle\right]_{n \times m}. \tag{18}$$

Step 2: Compute BFPRs $\mathfrak{B}_{\mathfrak{C}_k}; k = 1, 2, \cdots, m$ using formulas (6) and (7), respectively.

Step 3: Determine the BFPCs $[x_i]_{\mathfrak{B}_{C_k}}$ of alternatives x_i induced by each BFPR.

Step 4: Construct the BFPδ-nghds $^{C_k}\aleph^{\delta}_{x_i} = \left\langle {}^{C_k}\aleph^{\alpha}_{x_i}, {}^{C_k}\aleph^{\beta}_{x_i}\right\rangle$ of alternatives x_i corresponding to each critria C_k by using Definition 2.11.

Step 5: All the individual BFPδ-nghds can be transformed into a final aggregated BFPδ-nghd, which is given as:

$$\widetilde{\mathcal{N}^{\delta}_{x_i}} = \left\langle \widetilde{\mathcal{N}^{\alpha}_{x_i}}, \widetilde{\mathcal{N}^{\beta}_{x_i}}\right\rangle, \tag{19}$$

where

$$\widetilde{\mathcal{N}^{\alpha}_{x_i}} = \frac{\sum_{k=1}^{m} {}^{C_k}\aleph^{\alpha}_{x_i}}{m}, \tag{20}$$

and

$$\widetilde{\mathcal{N}^{\beta}_{x_i}} = \frac{\sum_{k=1}^{m} {}^{C_k}\aleph^{\beta}_{x_i}}{m}. \tag{21}$$

Step 6: According to Eq. (18), the best value $f_k^{\spadesuit} = \langle f_k^{\spadesuit^+}, f_k^{\spadesuit^-} \rangle$ and the worst value $f_k^{\bigstar} = \langle f_k^{\bigstar^+}, f_k^{\bigstar^-} \rangle$ can be determine against each criteria C_k as follows:

$$f_k^{\spadesuit} = \langle f_k^{\spadesuit^+}, f_k^{\spadesuit^-} \rangle = \left\{ \left\langle \bigvee_{i=1}^{n} \Psi_k^+(x_i), \bigwedge_{i=1}^{n} \Psi_k^-(x_i) \right\rangle : k = 1, 2, \cdots, m \right\}, \tag{22}$$

$$f_k^{\bigstar} = \langle f_k^{\bigstar^+}, f_k^{\bigstar^-} \rangle = \left\{ \left\langle \bigwedge_{i=1}^{n} \Psi_k^+(x_i), \bigvee_{i=1}^{n} \Psi_k^-(x_i) \right\rangle : k = 1, 2, \cdots, m \right\}. \tag{23}$$

Step 7: Calculate the maximum group utility value $\mathcal{S}_i = \left(\mathcal{S}_i^P, \mathcal{S}_i^N\right)$ and the minimum individual regret value $\mathcal{R}_i = \left(\mathcal{R}_i^P, \mathcal{R}_i^N\right)$ as follows:

$$\mathcal{S}_i^P = \sum_{k=1}^{n} \hbar_k \frac{d\left(f_k^{\spadesuit^+}, \Psi_k^+(x_i)\right)}{d\left(f_k^{\spadesuit^+}, f_k^{\bigstar^+}\right)}, \tag{24}$$

$$\mathcal{S}_i^N = \sum_{k=1}^{n} \hbar_k \frac{d\left(\Psi_k^-(x_i), f_k^{\bigstar^-}\right)}{d\left(f_k^{\bigstar^-}, f_k^{\spadesuit^-}\right)} \tag{25}$$

and

$$\mathcal{R}_i^P = \bigvee_i \left(\hbar_k \frac{d\left(f_k^{\spadesuit^+}, \Psi_k^+(x_i)\right)}{d\left(f_k^{\spadesuit^+}, f_k^{\bigstar^+}\right)} \right), \tag{26}$$

$$\mathcal{R}_i^N = \bigwedge_i \left(\hbar_k \frac{d\left(\Psi_k^-(x_i), f_k^{\bigstar^-}\right)}{d\left(f_k^{\bigstar^-}, f_k^{\spadesuit^-}\right)} \right). \tag{27}$$

Step 8: Further, by means of Definition 2.12, find the $BFP\delta C$ lower and upper approximations of utility measure and regret measure w.r.t. final aggregated BFPδ-nghd $\widetilde{\mathcal{N}_{x_i}^{\delta}}$, which are respectively given as:

$$\left.\begin{aligned}
\underline{(\mathcal{S}_i^P)(x)} &= \bigwedge_{y \in \Upsilon} \left\{ \left(1 - \widetilde{\mathcal{N}_{x_i}^{\alpha}}(y)\right) \vee \mathcal{S}_i^P(y) \right\}, \\
\underline{(\mathcal{S}_i^N)(x)} &= \bigvee_{y \in \Upsilon} \left\{ \widetilde{\mathcal{N}_{x_i}^{\beta}}(y) \wedge \mathcal{S}_i^N(y) \right\}, \\
\overline{(\mathcal{S}_i^P)(x)} &= \bigvee_{y \in \Upsilon} \left\{ \widetilde{\mathcal{N}_{x_i}^{\alpha}}(y) \wedge \mathcal{S}_i^P(y) \right\}, \\
\overline{(\mathcal{S}_i^N)(x)} &= \bigwedge_{y \in \Upsilon} \left\{ \left(-1 - \widetilde{\mathcal{N}_{x_i}^{\beta}}(y)\right) \vee \mathcal{S}_i^N(y) \right\}, \text{for every } x \in \Upsilon.
\end{aligned}\right\} \tag{28}$$

And

$$\left.\begin{aligned}\underline{(\mathcal{R}_i^P)(x)} &= \bigwedge_{y\in\Upsilon}\Big\{\Big(1-\widetilde{\mathcal{N}_{x_i}^{\alpha}}(y)\Big)\vee\mathcal{R}_i^P(y)\Big\},\\ \underline{(\mathcal{R}_i^N)(x)} &= \bigvee_{y\in\Upsilon}\Big\{\widetilde{\mathcal{N}_{x_i}^{\beta}}(y)\wedge\mathcal{R}_i^N(y)\Big\},\\ \overline{(\mathcal{R}_i^P)(x)} &= \bigvee_{y\in\Upsilon}\Big\{\widetilde{\mathcal{N}_{x_i}^{\alpha}}(y)\wedge\mathcal{R}_i^P(y)\Big\},\\ \overline{(\mathcal{R}_i^N)(x)} &= \bigwedge_{y\in\Upsilon}\Big\{\Big(-1-\widetilde{\mathcal{N}_{x_i}^{\beta}}(y)\Big)\vee\mathcal{R}_i^N(y)\Big\},\text{ for every } x\in\Upsilon.\end{aligned}\right\} \tag{29}$$

Step 9: Next, by utilizing the $BFP\delta C$ lower and upper approximations of $\mathcal{S}_i$ and $\mathcal{R}_i$, we find the lower and upper approximation vectors as follows:

$$\underline{\mathcal{S}_i} = \underline{\mathcal{S}_i^P} \oplus \underline{\mathcal{S}_i^N}, \tag{30}$$
$$\overline{\mathcal{S}_i} = \overline{\mathcal{S}_i^P} \oplus \overline{\mathcal{S}_i^N}, \tag{31}$$
$$\underline{\mathcal{R}_i} = \underline{\mathcal{R}_i^P} \oplus \underline{\mathcal{R}_i^N}, \tag{32}$$
$$\overline{\mathcal{R}_i} = \overline{\mathcal{R}_i^P} \oplus \overline{\mathcal{R}_i^N}, \tag{33}$$

Step 10: Find the values of $\mathcal{S}_i'$ and $\mathcal{R}_i'$ by using the following formulas:

$$\mathcal{S}_i' = \underline{\mathcal{S}_i} \oplus \overline{\mathcal{S}_i}, \tag{34}$$
$$\mathcal{R}_i' = \underline{\mathcal{R}_i} \oplus \overline{\mathcal{R}_i}. \tag{35}$$

Afterward, we determine the compromise $\mathcal{Q}_i$ for each alternatives by using the following formula:

$$\mathcal{Q}_i = \gamma\left(\frac{\mathcal{S}_i' - \mathcal{S}^*}{\mathcal{S}^- - \mathcal{S}^*}\right) + (1-\gamma)\left(\frac{\mathcal{R}_i' - \mathcal{R}^*}{\mathcal{R}^- - \mathcal{R}^*}\right), \tag{36}$$

where

$$\mathcal{S}^* = \bigwedge_{i=1}^{n}\mathcal{S}_i', \quad \mathcal{S}^- = \bigvee_{i=1}^{n}\mathcal{S}_i', \quad \mathcal{R}^* = \bigwedge_{i=1}^{n}\mathcal{R}_i', \quad \mathcal{R}^- = \bigvee_{i=1}^{n}\mathcal{R}_i';$$

and γ is served as a weight for the strategy of maximum group utility, while $1-\gamma$ is the weight of individual regret and its value falls in $[0,1]$. These strategies could be compromised (consensus) by $\gamma = 0.5$. The compromise solution can be chosen by majority $\gamma > 0.5$ and veto $\gamma < 0.5$.

The items are ranked by arranging the values of $\mathcal{S}_i'$, $\mathcal{R}_i'$, and $\mathcal{Q}_i$ in ascending order. The outcomes are three ranking lists w.r.t. the values of $\mathcal{S}_i'$, $\mathcal{R}_i'$, and $\mathcal{Q}_i$, which are further used to propose the compromise solution of alternatives. The term $\mathcal{Q}_i$ is the separation measure of alternative x_i from the optimal alternative, which shows that the minimum value of $\mathcal{Q}$ indicates the optimal alternative.

Lastly, we choose the optimal alternative x_1 as a compromise solution if the following two constraints are fulfilled:

Condition I (Tolerable advantage)

The first condition is illustrated as:

$$\mathcal{Q}\left(x^{(2)}\right)-\mathcal{Q}\left(x^{(1)}\right)\geq\frac{1}{n-1},\tag{37}$$

where $x^{(1)}$ and $x^{(2)}$ are high ranked two alternatives in $\mathcal{Q}_i$ and n is the number of alternatives.

Condition II (Tolerable stability)

In this condition the alternative $x^{(1)}$ should also best ranked alternative in $\mathcal{S}'_i$ and $\mathcal{R}'_i$.

If the above two conditions do not hold together, then there are many compromise solutions, which are attained as follows:

1. If the only *condition II* is not fulfilled then the set of compromise solutions comprises alternatives $x^{(1)}$ and $x^{(2)}$.
2. If the only *condition I* is not satisfied then the compromise solution consists of alternatives $x^{(1)}, x^{(2)}, \cdots, x^{(N)}$, where $x^{(N)}$ is determine by the relation:

$$\mathcal{Q}\left(x^{(N)}\right)-\mathcal{Q}\left(x^{(1)}\right)<\frac{1}{n-1};\text{ for maximum } N.\tag{38}$$

In light of the above discussion, the procedure of the devised VIKOR method under the BFPδC-BFRS model is portrayed in the following algorithm. Moreover, Figure 1 depicts the graphical illustration of the recommended MCDM mechanism.

Algorithm 1: An algorithm for the MCDM problem

Input: Bipolar fuzzy information system $(\Upsilon, C, \mathfrak{W}, \mathcal{E})$.
Output: A ranking result of all alternatives.
Step 1: Determine $\mathfrak{B}_{C_k}; k=1,2,\cdots,m$ using formulation exhibited in Eqs. (6) and (7).
Step 2: Compute $[x_i]_{\mathfrak{B}_{C_k}}$ of x_i regarding C_k.
Step 3: Compute ${}^{C_k}\aleph^{\delta}_{x_i}=\left\langle {}^{C_k}\aleph^{\alpha}_{x_i}, {}^{C_k}\aleph^{\beta}_{x_i}\right\rangle$ of x_i regarding C_k.
Step 4: Determine aggregated BFPδ-nghd $\widetilde{\mathcal{N}^{\delta}_{x_i}}$ using Eqs. (19), (20) and (21).
Step 5: Compute the best and the wrest values against each criterion C_k according to Eqs. (22) and (23).
Step 6: Calculate $\mathcal{S}_i=\left(\mathcal{S}^P_i,\mathcal{S}^N_i\right)$ and $\mathcal{R}_i=\left(\mathcal{R}^P_i,\mathcal{R}^N_i\right)$ according to Eqs. (24), (25), (26) and (27).
Step 7: Find the $BFP\delta C$ lower and upper approximations of $\mathcal{S}_i$ and $\mathcal{R}_i$ w.r.t. $\widetilde{\mathcal{N}^{\delta}_{x_i}}$ using Eqs. (28) and (29).
Step 8: Calculate the lower and upper approximation vectors.
Step 9: Compute the values of $\mathcal{S}'_i$, $\mathcal{R}'_i$ and $\mathcal{Q}_i$ according to Eq. (36).
Step 10: Rank the items in terms of $\mathcal{S}'_i$, $\mathcal{R}'_i$ and $\mathcal{Q}_i$.

4. Application of the Proposed VIKOR Approach

In this part, a real-world decision-making issue employing bipolar fuzzy information has been utilized to analyze the developed approach.

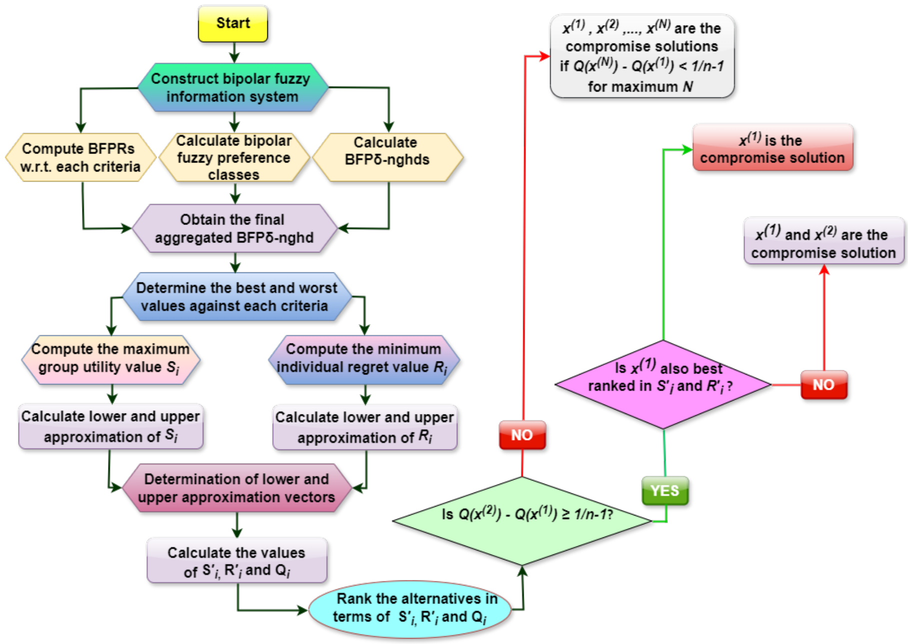

Fig. 1. Flowchart portrayal of the projected of the approach

Imagine a decision-maker who is having trouble choosing between the five smartphones that are on display for sale in a store. Since it is well known that each feature influences the price and usefulness of the desired smartphone, let's presume that he is concentrating on the subsequent features (criteria) to choose the finest smartphone: colour, memory, elegancy, and camera zoom. Therefore, the attributes listed above stand in for the requirements in our MCDM issue, and these smartphones represent the alternatives.

Let $C = \{C_1, C_2, C_3, C_4\}$ be the set of concerned features, and $\Upsilon = \{x_1, x_2, x_3, x_4, x_5\}$ be the collection of concerned smartphones. All the criteria weights are provided as follows: $\mathfrak{W} = (0.25, 0.3, 0.25, 0.2)^T$. Table 2 shows the alternatives' ratings according to the criteria.

Table 2
Bipolar fuzzy information system

Υ/C	C_1	C_2	C_3	C_4
x_1	(0.5, - 0.25)	(0.8, - 0.7)	(0.3, - 0.1)	(0.6, - 0.6)
x_2	(0.2, - 0.8)	(0.9, - 0.4)	(0.6, - 0.3)	(0.55, - 0.5)
x_3	(0.33, - 0.25)	(0.75, - 0.4)	(0.25, - 0.7)	(0.3, - 0.1)
x_4	(0.65, - 0.6)	(0.3, - 0.75)	(0.8, - 0.35)	(0.65, - 0.7)
x_5	(1, - 0.5)	(0.4, - 0.35)	(0.2, - 0.6)	(0.25, - 0.65)

The following computation phases are displayed to help choose the best smartphone:

Step 1: Based on criteria C_1, C_2, C_3, C_4 and according to formulas (6) and (7) to compute the BFPRs of alternative x_i to the alternative $x_j(i,j=1,2,\cdots,5)$, we get:

$$\mathfrak{B}_{C_1}(x_i,x_j)=\begin{pmatrix}\langle 0.500,-0.500\rangle & \langle 0.650,-0.775\rangle & \langle 0.585,-0.500\rangle & \langle 0.425,-0.675\rangle & \langle 0.250,-0.625\rangle\\ \langle 0.350,-0.225\rangle & \langle 0.500,-0.500\rangle & \langle 0.435,-0.225\rangle & \langle 0.275,-0.400\rangle & \langle 0.100,-0.350\rangle\\ \langle 0.415,-0.500\rangle & \langle 0.565,-0.775\rangle & \langle 0.500,-0.500\rangle & \langle 0.340,-0.675\rangle & \langle 0.165,-0.625\rangle\\ \langle 0.575,-0.325\rangle & \langle 0.725,-0.600\rangle & \langle 0.660,-0.325\rangle & \langle 0.500,-0.500\rangle & \langle 0.325,-0.450\rangle\\ \langle 0.750,-0.375\rangle & \langle 0.900,-0.650\rangle & \langle 0.835,-0.375\rangle & \langle 0.675,-0.550\rangle & \langle 0.500,-0.500\rangle\end{pmatrix}, \tag{39}$$

$$\mathfrak{B}_{C_2}(x_i,x_j)=\begin{pmatrix}\langle 0.500,-0.500\rangle & \langle 0.450,-0.350\rangle & \langle 0.525,-0.350\rangle & \langle 0.750,-0.525\rangle & \langle 0.700,-0.325\rangle\\ \langle 0.550,-0.650\rangle & \langle 0.500,-0.500\rangle & \langle 0.575,-0.500\rangle & \langle 0.800,-0.675\rangle & \langle 0.750,-0.475\rangle\\ \langle 0.475,-0.650\rangle & \langle 0.425,-0.500\rangle & \langle 0.500,-0.500\rangle & \langle 0.725,-0.675\rangle & \langle 0.675,-0.475\rangle\\ \langle 0.250,-0.475\rangle & \langle 0.200,-0.325\rangle & \langle 0.275,-0.325\rangle & \langle 0.500,-0.500\rangle & \langle 0.450,-0.300\rangle\\ \langle 0.300,-0.675\rangle & \langle 0.250,-0.525\rangle & \langle 0.325,-0.525\rangle & \langle 0.550,-0.700\rangle & \langle 0.500,-0.500\rangle\end{pmatrix}, \tag{40}$$

$$\mathfrak{B}_{C_3}(x_i,x_j)=\begin{pmatrix}\langle 0.500,-0.500\rangle & \langle 0.350,-0.600\rangle & \langle 0.525,-0.800\rangle & \langle 0.250,-0.625\rangle & \langle 0.550,-0.750\rangle\\ \langle 0.650,-0.400\rangle & \langle 0.500,-0.500\rangle & \langle 0.675,-0.700\rangle & \langle 0.400,-0.525\rangle & \langle 0.700,-0.650\rangle\\ \langle 0.475,-0.200\rangle & \langle 0.325,-0.300\rangle & \langle 0.500,-0.500\rangle & \langle 0.225,-0.325\rangle & \langle 0.525,-0.450\rangle\\ \langle 0.750,-0.375\rangle & \langle 0.600,-0.475\rangle & \langle 0.775,-0.675\rangle & \langle 0.500,-0.500\rangle & \langle 0.800,-0.625\rangle\\ \langle 0.450,-0.250\rangle & \langle 0.300,-0.350\rangle & \langle 0.475,-0.550\rangle & \langle 0.200,-0.375\rangle & \langle 0.500,-0.500\rangle\end{pmatrix}, \tag{41}$$

$$\mathfrak{B}_{C_4}(x_i,x_j)=\begin{pmatrix}\langle 0.500,-0.500\rangle & \langle 0.525,-0.450\rangle & \langle 0.650,-0.250\rangle & \langle 0.475,-0.550\rangle & \langle 0.675,-0.525\rangle\\ \langle 0.475,-0.550\rangle & \langle 0.500,-0.500\rangle & \langle 0.625,-0.300\rangle & \langle 0.450,-0.600\rangle & \langle 0.650,-0.575\rangle\\ \langle 0.350,-0.750\rangle & \langle 0.375,-0.700\rangle & \langle 0.500,-0.500\rangle & \langle 0.325,-0.800\rangle & \langle 0.525,-0.775\rangle\\ \langle 0.525,-0.450\rangle & \langle 0.550,-0.400\rangle & \langle 0.675,-0.200\rangle & \langle 0.500,-0.500\rangle & \langle 0.700,-0.475\rangle\\ \langle 0.325,-0.475\rangle & \langle 0.350,-0.425\rangle & \langle 0.475,-0.225\rangle & \langle 0.300,-0.525\rangle & \langle 0.500,-0.500\rangle\end{pmatrix}. \tag{42}$$

Step 2: The BFPCs $[x_i]_{\mathfrak{B}_{C_1}}$, $[x_i]_{\mathfrak{B}_{C_2}}$, $[x_i]_{\mathfrak{B}_{C_3}}$ and $[x_i]_{\mathfrak{B}_{C_4}}$ are respectively listed in Tables 3, 4, 5 and 6.

From Tables 3, 4, 5 and 6, we can observe that $P\left(\mathfrak{B}_{\mathfrak{C}_k}\right)=\left\{[x_i]_{\mathfrak{B}_{\mathfrak{C}_k}}: i=1,2,\cdots,5, k=1,2,3,4\right\}$ is a $BFP\delta C$ of Υ $\left(\delta=\langle 0.500,-0.500\rangle\right)$.

Step 3: Let $\delta=\langle\alpha,\beta\rangle=\langle 0.500,-0.500\rangle$ be the critical value. Then the elements ${}^{C_k}\aleph^{\delta}_{x_i}=\left\langle {}^{C_k}\aleph^{\alpha}_{x_i}, {}^{C_k}\aleph^{\beta}_{x_i}\right\rangle$ $(i=1,2\cdots,5,k=1,2,3,4)$ are displayed in Tables 7, 8, 9 and 10.

Step 4: The aggregated BFPδ-nghd $\widetilde{\mathcal{N}^{\delta}}_{x_i}$ is given in Table 11 as follows:

Step 5: The best and the wrest values against each criteria C_k can be evaluated according to Eqs. (22) and (23), respectively, which are shown in Table 12.

Table 3

The BFPSc $[x_i]_{\mathfrak{B}_{C_1}}$

	$[x_1]_{\mathfrak{B}_{C_1}}$	$[x_2]_{\mathfrak{B}_{C_1}}$	$[x_3]_{\mathfrak{B}_{C_1}}$	$[x_4]_{\mathfrak{B}_{C_1}}$	$[x_5]_{\mathfrak{B}_{C_1}}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.350, -0.225\rangle$	$\langle 0.415, -0.500\rangle$	$\langle 0.575, -0.325\rangle$	$\langle 0.750, -0.375\rangle$
x_2	$\langle 0.650, -0.775\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.565, -0.775\rangle$	$\langle 0.725, -0.600\rangle$	$\langle 0.900, -0.650\rangle$
x_3	$\langle 0.585, -0.500\rangle$	$\langle 0.435, -0.225\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.660, -0.325\rangle$	$\langle 0.835, -0.375\rangle$
x_4	$\langle 0.425, -0.675\rangle$	$\langle 0.275, -0.400\rangle$	$\langle 0.340, -0.675\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.675, -0.550\rangle$
x_5	$\langle 0.250, -0.625\rangle$	$\langle 0.100, -0.350\rangle$	$\langle 0.165, -0.625\rangle$	$\langle 0.325, -0.450\rangle$	$\langle 0.500, -0.500\rangle$

Table 4

The BFPCs $[x_i]_{\mathfrak{B}_{C_2}}$

	$[x_1]_{\mathfrak{B}_{C_2}}$	$[x_2]_{\mathfrak{B}_{C_2}}$	$[x_3]_{\mathfrak{B}_{C_2}}$	$[x_4]_{\mathfrak{B}_{C_2}}$	$[x_5]_{\mathfrak{B}_{C_2}}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.550, -0.650\rangle$	$\langle 0.475, -0.650\rangle$	$\langle 0.250, -0.475\rangle$	$\langle 0.300, -0.675\rangle$
x_2	$\langle 0.450, -0.350\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.425, -0.500\rangle$	$\langle 0.200, -0.325\rangle$	$\langle 0.250, -0.525\rangle$
x_3	$\langle 0.525, -0.350\rangle$	$\langle 0.575, -0.500\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.275, -0.325\rangle$	$\langle 0.325, -0.525\rangle$
x_4	$\langle 0.750, -0.525\rangle$	$\langle 0.800, -0.675\rangle$	$\langle 0.725, -0.675\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.550, -0.700\rangle$
x_5	$\langle 0.700, -0.325\rangle$	$\langle 0.750, -0.475\rangle$	$\langle 0.675, -0.475\rangle$	$\langle 0.450, -0.300\rangle$	$\langle 0.500, -0.500\rangle$

Table 5

The BFPCs $[x_i]_{\mathfrak{B}_{C_3}}$

	$[x_1]_{\mathfrak{B}_{C_3}}$	$[x_2]_{\mathfrak{B}_{C_3}}$	$[x_3]_{\mathfrak{B}_{C_3}}$	$[x_4]_{\mathfrak{B}_{C_3}}$	$[x_5]_{\mathfrak{B}_{C_3}}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.650, -0.400\rangle$	$\langle 0.475, -0.200\rangle$	$\langle 0.750, -0.375\rangle$	$\langle 0.450, -0.250\rangle$
x_2	$\langle 0.350, -0.600\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.325, -0.300\rangle$	$\langle 0.600, -0.475\rangle$	$\langle 0.300, -0.350\rangle$
x_3	$\langle 0.525, -0.800\rangle$	$\langle 0.675, -0.700\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.775, -0.675\rangle$	$\langle 0.475, -0.550\rangle$
x_4	$\langle 0.250, -0.625\rangle$	$\langle 0.400, -0.525\rangle$	$\langle 0.225, -0.325\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.200, -0.375\rangle$
x_5	$\langle 0.550, -0.750\rangle$	$\langle 0.700, -0.650\rangle$	$\langle 0.525, -0.450\rangle$	$\langle 0.800, -0.625\rangle$	$\langle 0.500, -0.500\rangle$

Table 6

The BFPCs $[x_i]_{\mathfrak{B}_{C_4}}$

	$[x_1]_{\mathfrak{B}_{C_4}}$	$[x_2]_{\mathfrak{B}_{C_4}}$	$[x_3]_{\mathfrak{B}_{C_4}}$	$[x_4]_{\mathfrak{B}_{C_4}}$	$[x_5]_{\mathfrak{B}_{C_4}}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.475, -0.550\rangle$	$\langle 0.350, -0.750\rangle$	$\langle 0.525, -0.450\rangle$	$\langle 0.325, -0.475\rangle$
x_2	$\langle 0.525, -0.450\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.375, -0.700\rangle$	$\langle 0.550, -0.400\rangle$	$\langle 0.350, -0.425\rangle$
x_3	$\langle 0.650, -0.250\rangle$	$\langle 0.625, -0.300\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.675, -0.200\rangle$	$\langle 0.475, -0.225\rangle$
x_4	$\langle 0.475, -0.550\rangle$	$\langle 0.450, -0.600\rangle$	$\langle 0.325, -0.800\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.300, -0.525\rangle$
x_5	$\langle 0.675, -0.525\rangle$	$\langle 0.650, -0.575\rangle$	$\langle 0.525, -0.775\rangle$	$\langle 0.700, -0.475\rangle$	$\langle 0.500, -0.500\rangle$

Step 6: The values of utility measure $\mathcal{S}_i = \left(\mathcal{S}_i^+, \mathcal{S}_i^-\right)$ and regret measure $\mathcal{R}_i = \left(\mathcal{R}_i^+, \mathcal{R}_i^-\right)$ are evaluated by using Eqs. (24), (25), (26) and (27), respectively, and the outcomes are displayed in Table 13.

Step 7: Using Eqs. (28) and (29), the $BFP\delta C$ lower and upper approximations of $\mathcal{S}_i$ and $\mathcal{R}_i$ regarding $\widetilde{\mathcal{N}_{x_i}^{\delta}}$ are

Table 7
The BFPδ-nghd ${}^{C_1}\aleph^{\delta}_{x_i}$

	${}^{C_1}\aleph^{\delta}_{x_1}$	${}^{C_1}\aleph^{\delta}_{x_2}$	${}^{C_1}\aleph^{\delta}_{x_3}$	${}^{C_1}\aleph^{\delta}_{x_4}$	${}^{C_1}\aleph^{\delta}_{x_5}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.350, -0.225\rangle$	$\langle 0.415, -0.500\rangle$	$\langle 0.575, -0.325\rangle$	$\langle 0.750, -0.375\rangle$
x_2	$\langle 0.650, -0.775\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.565, -0.775\rangle$	$\langle 0.725, -0.600\rangle$	$\langle 0.900, -0.650\rangle$
x_3	$\langle 0.585, -0.500\rangle$	$\langle 0.435, -0.225\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.660, -0.325\rangle$	$\langle 0.835, -0.375\rangle$
x_4	$\langle 0.425, -0.675\rangle$	$\langle 0.275, -0.400\rangle$	$\langle 0.340, -0.675\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.675, -0.550\rangle$
x_5	$\langle 0.250, -0.625\rangle$	$\langle 0.100, -0.350\rangle$	$\langle 0.165, -0.625\rangle$	$\langle 0.325, -0.450\rangle$	$\langle 0.500, -0.500\rangle$

Table 8
The BFPδ-nghd ${}^{C_2}\aleph^{\delta}_{x_i}$

	${}^{C_2}\aleph^{\delta}_{x_1}$	${}^{C_2}\aleph^{\delta}_{x_2}$	${}^{C_2}\aleph^{\delta}_{x_3}$	${}^{C_2}\aleph^{\delta}_{x_4}$	${}^{C_2}\aleph^{\delta}_{x_5}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.550, -0.650\rangle$	$\langle 0.475, -0.650\rangle$	$\langle 0.250, -0.475\rangle$	$\langle 0.300, -0.675\rangle$
x_2	$\langle 0.450, -0.350\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.425, -0.500\rangle$	$\langle 0.200, -0.325\rangle$	$\langle 0.250, -0.525\rangle$
x_3	$\langle 0.525, -0.350\rangle$	$\langle 0.575, -0.500\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.275, -0.325\rangle$	$\langle 0.325, -0.525\rangle$
x_4	$\langle 0.750, -0.525\rangle$	$\langle 0.800, -0.675\rangle$	$\langle 0.725, -0.675\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.550, -0.700\rangle$
x_5	$\langle 0.700, -0.325\rangle$	$\langle 0.750, -0.475\rangle$	$\langle 0.675, -0.475\rangle$	$\langle 0.450, -0.300\rangle$	$\langle 0.500, -0.500\rangle$

Table 9
The BFPδ-nghd ${}^{C_3}\aleph^{\delta}_{x_i}$

	${}^{C_3}\aleph^{\delta}_{x_1}$	${}^{C_3}\aleph^{\delta}_{x_2}$	${}^{C_3}\aleph^{\delta}_{x_3}$	${}^{C_3}\aleph^{\delta}_{x_4}$	${}^{C_3}\aleph^{\delta}_{x_5}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.650, -0.400\rangle$	$\langle 0.475, -0.200\rangle$	$\langle 0.750, -0.375\rangle$	$\langle 0.450, -0.250\rangle$
x_2	$\langle 0.350, -0.600\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.325, -0.300\rangle$	$\langle 0.600, -0.475\rangle$	$\langle 0.300, -0.350\rangle$
x_3	$\langle 0.525, -0.800\rangle$	$\langle 0.675, -0.700\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.775, -0.675\rangle$	$\langle 0.475, -0.550\rangle$
x_4	$\langle 0.250, -0.625\rangle$	$\langle 0.400, -0.525\rangle$	$\langle 0.225, -0.325\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.200, -0.375\rangle$
x_5	$\langle 0.550, -0.750\rangle$	$\langle 0.700, -0.650\rangle$	$\langle 0.525, -0.450\rangle$	$\langle 0.800, -0.625\rangle$	$\langle 0.500, -0.500\rangle$

Table 10
The BFPδ-nghd ${}^{C_4}\aleph^{\delta}_{x_i}$

	${}^{C_4}\aleph^{\delta}_{x_1}$	${}^{C_4}\aleph^{\delta}_{x_2}$	${}^{C_4}\aleph^{\delta}_{x_3}$	${}^{C_4}\aleph^{\delta}_{x_4}$	${}^{C_4}\aleph^{\delta}_{x_5}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.475, -0.550\rangle$	$\langle 0.350, -0.750\rangle$	$\langle 0.525, -0.450\rangle$	$\langle 0.325, -0.475\rangle$
x_2	$\langle 0.525, -0.450\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.375, -0.700\rangle$	$\langle 0.550, -0.400\rangle$	$\langle 0.350, -0.425\rangle$
x_3	$\langle 0.650, -0.250\rangle$	$\langle 0.625, -0.300\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.675, -0.200\rangle$	$\langle 0.475, -0.225\rangle$
x_4	$\langle 0.475, -0.550\rangle$	$\langle 0.450, -0.600\rangle$	$\langle 0.325, -0.800\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.300, -0.525\rangle$
x_5	$\langle 0.675, -0.525\rangle$	$\langle 0.650, -0.575\rangle$	$\langle 0.525, -0.775\rangle$	$\langle 0.700, -0.475\rangle$	$\langle 0.500, -0.500\rangle$

respectively given in Table 14.

Step 8: We find the lower and upper approximation vectors using Eqs. (30 - 33) and the results are listed in Table 15 as follows:

Table 11
The aggregated BFPδ-nghd $\widetilde{\mathcal{N}^{\delta}_{x_i}}$

	$\widetilde{\mathcal{N}^{\delta}_{x_1}}$	$\widetilde{\mathcal{N}^{\delta}_{x_2}}$	$\widetilde{\mathcal{N}^{\delta}_{x_3}}$	$\widetilde{\mathcal{N}^{\delta}_{x_4}}$	$\widetilde{\mathcal{N}^{\delta}_{x_5}}$
x_1	$\langle 0.500, -0.500\rangle$	$\langle 0.506, -0.456\rangle$	$\langle 0.429, -0.525\rangle$	$\langle 0.525, -0.406\rangle$	$\langle 0.456, -0.444\rangle$
x_2	$\langle 0.494, -0.544\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.423, -0.569\rangle$	$\langle 0.519, -0.450\rangle$	$\langle 0.450, -0.488\rangle$
x_3	$\langle 0.571, -0.475\rangle$	$\langle 0.578, -0.431\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.596, -0.381\rangle$	$\langle 0.528, -0.419\rangle$
x_4	$\langle 0.475, -0.594\rangle$	$\langle 0.481, -0.550\rangle$	$\langle 0.404, -0.619\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.431, -0.538\rangle$
x_5	$\langle 0.544, -0.556\rangle$	$\langle 0.550, -0.513\rangle$	$\langle 0.473, -0.581\rangle$	$\langle 0.569, -0.463\rangle$	$\langle 0.500, -0.500\rangle$

Table 12
The values of f_k^+ and f_k^-

	C_1	C_2	C_3	C_4
f_k^+	$\langle 1, -0.8\rangle$	$\langle 0.9, -0.75\rangle$	$\langle 0.8, -0.7\rangle$	$\langle 0.65, -0.7\rangle$
f_k^-	$\langle 0.2, -0.25\rangle$	$\langle 0.3, -0.35\rangle$	$\langle 0.2, -0.1\rangle$	$\langle 0.25, -0.1\rangle$

Table 13
The values of $\mathcal{S}_i$ and $\mathcal{R}_i$

	x_1	x_2	x_3	x_4	x_5
$\mathcal{S}_i$	$\langle 0.4396, -0.4667\rangle$	$\langle 0.3833, -0.5095\rangle$	$\langle 0.6885, -0.2929\rangle$	$\langle 0.4094, -0.8061\rangle$	$\langle 0.7000, -0.5053\rangle$
$\mathcal{R}_i$	$\langle 0.1562, -0.3000\rangle$	$\langle 0.2500, -0.2500\rangle$	$\langle 0.2094, -0.2500\rangle$	$\langle 0.3000, -0.3429\rangle$	$\langle 0.2500, -0.2083\rangle$

Table 14
The lower and upper approximations of $\mathcal{S}_i$ and $\mathcal{R}_i$

	x_1	x_2	x_3	x_4	x_5
$(\underline{\mathcal{S}_i^P}, \underline{\mathcal{S}_i^N})$	$\langle 0.425, -0.475\rangle$	$\langle 0.494, -0.431\rangle$	$\langle 0.571, -0.500\rangle$	$\langle 0.475, -0.381\rangle$	$\langle 0.544, -0.419\rangle$
$(\overline{\mathcal{S}_i^P}, \overline{\mathcal{S}_i^N})$	$\langle 0.571, -0.4667\rangle$	$\langle 0.578, -0.500\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.596, -0.5095\rangle$	$\langle 0.528, -0.5095\rangle$
$(\underline{\mathcal{R}_i^P}, \underline{\mathcal{R}_i^N})$	$\langle 0.429, -0.475\rangle$	$\langle 0.422, -0.431\rangle$	$\langle 0.500, -0.500\rangle$	$\langle 0.404, -0.381\rangle$	$\langle 0.472, -0.419\rangle$
$(\overline{\mathcal{R}_i^P}, \overline{\mathcal{R}_i^N})$	$\langle 0.300, -0.3429\rangle$	$\langle 0.300, -0.3429\rangle$	$\langle 0.300, -0.3429\rangle$	$\langle 0.300, -0.3429\rangle$	$\langle 0.300, -0.3429\rangle$

Table 15
Values of $\underline{\mathcal{S}_i}$, $\overline{\mathcal{S}_i}$, $\underline{\mathcal{R}_i}$ and $\overline{\mathcal{S}_i}$

	x_1	x_2	x_3	x_4	x_5
$\underline{\mathcal{S}_i}$	-0.05	0.063	0.071	0.094	0.125
$\overline{\mathcal{S}_i}$	0.1043	0.078	0	0.0865	0.0185
$\underline{\mathcal{R}_i}$	-0.046	-0.009	0	0.023	0.053
$\overline{\mathcal{R}_i}$	-0.0429	-0.0429	-0.0429	-0.0429	-0.0429

Step 9: Finally, by employing Eqs. (34), (35) and (36), the outputs of $\mathcal{S}'_i$, $\mathcal{R}'_i$ and $\mathcal{Q}_i$ for $\gamma = 0.5$ are determined in Table 16. By taking the ascending order of $\mathcal{S}'_i$, $\mathcal{R}'_i$ and $\mathcal{Q}_i$, we acquired three distinct ranking outcomes of five alternatives which are displayed in Table 16. These rankings are pictorially depicted in Figure 2.

In light of Table 16, it becomes evident that x1 is the optimal alternative for all three ranking lists.

Table 16
Values of $\mathcal{S}_i'$, $\mathcal{R}_i'$ and $\mathcal{Q}_i$

	x_1	x_2	x_3	x_4	x_5	Ranking
$\mathcal{S}_i'$	0.0543	0.141	0.071	0.1805	0.1435	$x_1 \succeq x_3 \succeq x_2 \succeq x_5 \succeq x_4$
$\mathcal{R}_i'$	-0.0889	0.519	-0.0429	-0.0199	0.0101	$x_1 \succeq x_3 \succeq x_4 \succeq x_5 \succeq x_2$
$\mathcal{Q}_i$	0	0.8435	0.1040	0.5568	0.4348	$x_1 \succeq x_3 \succeq x_5 \succeq x_4 \succeq x_2$

But,

$$\mathcal{Q}\left(x^{(2)}\right) - \mathcal{Q}\left(x^{(1)}\right) = \mathcal{Q}\left(x_3\right) - \mathcal{Q}\left(x_1\right) = 0.1040 < \frac{1}{5-1} = 0.25.$$

Hence, the proposed algorithm's first condition (tolerable advantage) is not justified. So, $\{x_1, x_3\}$ is a set of compromise solutions.

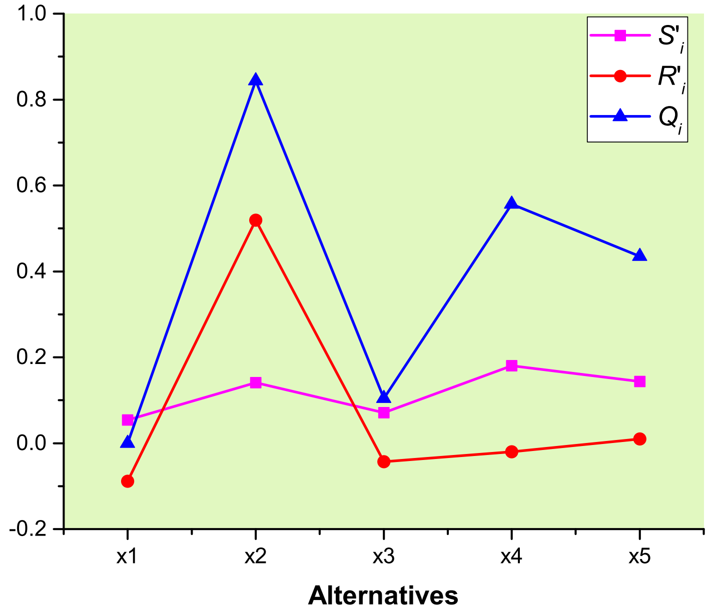

Fig. 2. Ranking of alternatives

5. Comparative study and Discussion

The benefits of the suggested methodology are examined in this section, along with a comparison with existing methods. We compare our devised scheme with some prevalent approaches in the literature including, Malik and Shabir [24], Wei *et al.* [27], Jana *et al.* [28], Gul [29], and Gul *et al.* [30]. These comparison outcomes are displayed in Table 17.

According to 17, we can see that the ranking outcomes of alternatives of the other approaches are different, mainly due to alterations in the decision-making context. Yet, the optimal choice is identical. This fact is common in decision analysis. Generally, decision-makers can allocate various inputs of α and β according to their preferences and real demands.

Table 17
Comparison with some existing methods

Methods	Ranking of alternatives	Optimal alternative
Malik and Shabir [24]	$x_1 \succeq x_2 \succeq x_3 \succeq x_4 \succeq x_5$	x_1
Wei *et al.* [27]	$x_1 \succeq x_2 \succeq x_3 \succeq x_4 \succeq x_5$	x_1
Jana *et al.* [28]	$x_1 \succeq x_3 \succeq x_4 \succeq x_2 \succeq x_5$	x_1
Gul [29]	$x_1 \succeq x_2 \succeq x_3 \succeq x_4 \succeq x_5$	x_1
Gul *et al.* [30]	$x_1 \approx x_4 \succeq x_3 \succeq x_5 \succeq x_2$	x_1, x_4
Our proposed approach	$x_1 \succeq x_3 \succeq x_5 \succeq x_4 \succeq x_2$	x_1

5.1 *Merits of the devised strategy*

To illustrate the uniqueness and superiority of our suggested strategy, we go over the drawbacks of current approaches and how the recommended framework addresses these issues.

1. MCDM issues with bipolar fuzzy data are studied using a variety of BFR-based decision-making techniques. But, not every MADM issue can be described by a BFR. Because of this, we provide an approach based on the BFPδC-BFRS variant for tackling MCDM issues using bipolar fuzzy information.

2. Comparing our proposed method to the approaches described in [1, 10–15], we find that these approaches are unable to adequately reflect bipolarity in the decision procedure, which is a crucial aspect of human perception and behaviour.

3. Fuzzy decision-making strategies are substantially adopted to address issues with merely one-sided data; i.e., objects are ranked via the PMG. By using fuzzy structure in decision-making, we are unable to offer details about the dissatisfaction degree of alternatives regarding various criteria. Therefore, we advocate a BFPδC-BFRSs framework to rank the items.

5.2 *Limitations*

The framed scheme highly relies on the adequate input of the parameters α and β. Selecting the best items is difficult and frequently arbitrary. In practice, various inputs of α and β may provide varied ranking outcomes, making the technique less reliable.

6. Conclusions

FSs and RSs have several practical applications and are efficient mathematical techniques for circumventing uncertainty. Meanwhile, BFSs address both fuzziness and bipolarity simultaneously. In the MCDM technique, the alternatives are compared against each other based on their relative performance to each other. Furthermore, to ascertain the relative importance of each criterion, the VIKOR technique necessitates a comparison of the criteria. The best compromise solution is determined by the highest ranking of the alternatives. In this script, we established an innovative hybrid VIKOR method within the context of the BFPδC-BFRSs. The proficiency and applicability of the framed methodology have been highlighted using a numerical example. Finally, we conducted a comparison of the devised methodology with some other prevailing approaches and analyzed how our framed approach is superior to existing ones.

In the future, we extend our to more generalized frameworks including covering-based (α, β)-multi-granulation bipolar FRS model [30], and m-polar FSs [31].

Conflicts of Interest

The authors declare no conflicts of interest.

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