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# Some Refinements of Integral Inequalities over Triangular Fuzzy Co-Domain

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**ABSTRACT**

Integral inequalities, in general, serve as powerful tools for various applications. Specifically, when an integral operator is used as a predictive tool, an integral inequality can play a key role in defining, quantifying, and analyzing such processes. Real-valued functions over a fuzzy domain, also referred to as real-valued fuzzy functions, offer a valuable approach for incorporating uncertainty into prediction models. In this paper, using a straightforward proof method over a newly defined triangular $L_P$fuzzy space, we establish several new refinements for integral forms of the classical Hölder's and newly defined triangular Hölder's-like inequality. Numerous existing inequalities linked with the triangular Hölder's-like inequality over a fuzzy domain can be improved through the newly obtained ones, as illustrated through applications such as the triangular Hölder's power-mean-like integral inequality, triangular Cauchy–Schwarz-like inequality, triangular Minkowski's-like inequality, and triangular Beckenbach's-like inequality over a fuzzy domain. Additionally, our outcomes represent significant progressions in the field of mathematics.

## 1. Introduction

The theory of inequalities is extensively taught due to its numerous applications across various scientific and technical fields. Mathematical inequalities are utilized in system design, engineering, signal processing, and optimization problems. They provide a robust framework for analyzing and understanding the behavior of solutions to numerical and partial differential equations. Within this framework, fractional analysis serves as an innovative extension of classical analysis to non-integer orders. Fractional integral inequalities are particularly useful in studying physical systems governed by fractional differential equations, helping to derive energy estimates, as demonstrated in [1–7]. Convex functions have been a significant focus of mathematical research for a considerable time. Their growing application in machine learning, optimization, and various scientific and technical disciplines likely contributes to their increasing prominence [8–12]. There exists a profound and intricate connection between convex functions and the theory of inequalities. Convex functions have

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played a pivotal role in the formulation and discovery of numerous important and practical inequalities. A convex function on convex sets is defined as follows.

If $\Psi: K \to \mathfrak{R}$ is a convex function defined on the interval $K$ of real numbers, and $\theta, \lambda \in K$ with $\theta < \lambda$, then

$$\Psi(\kappa\theta + (1-\kappa)\lambda) \leq \kappa\Psi(\theta) + (1-\kappa)\Psi(\lambda) \tag{1}$$

Jensen's inequality, along with its related forms and extensions, is a fundamental and widely recognized result for convex functions. Its broad applicability highlights its importance across various scientific, engineering, and computational fields (refer to [13, 14]). Convex functions serve as essential tools in contemporary research and form the basis of many mathematical inequalities.

In this regard, we are reminded of the well-known inequity resulting from Hermite and Hadamard acting independently.

Theorem 1: Assume that the convex mapping $\Psi: [\mathfrak{v}, \mathfrak{y}] \to \mathfrak{R}$. Then, the following double-inequality holds:

$$\Psi\left(\frac{\mathfrak{v}+\mathfrak{y}}{2}\right) \leq \frac{1}{\mathfrak{y}-\mathfrak{v}} \int_{\mathfrak{v}}^{\mathfrak{y}} \Psi(x)dx \leq \frac{\Psi(\mathfrak{y})+\Psi(\mathfrak{v})}{2} \tag{2}$$

where $\mathfrak{R}$ is set of real number.

With the purpose of generalizing, enhancing, and expanding upon the well-known integral inequality (1), see [15].

Suppose $\mathfrak{p} > 1$ and $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. If $\Psi$ and $\mathcal{E}$ are two real functions defined on $[\mathfrak{v}, \mathfrak{y}]$ such that $|\Psi|^{\mathfrak{p}}$ and $|\mathcal{E}|^{\mathfrak{q}}$ are integrable functions on $[\mathfrak{v}, \mathfrak{y}]$, then,

$$\int_{\mathfrak{v}}^{\mathfrak{y}} |\Psi(\mathfrak{o})\mathcal{E}(\mathfrak{o})| \, d\mathfrak{o} \leq \left[\int_{\mathfrak{v}}^{\mathfrak{y}} |\Psi(\mathfrak{o})|^{\mathfrak{p}} \, d\mathfrak{o}\right]^{\frac{1}{\mathfrak{p}}} \left[\int_{\mathfrak{v}}^{\mathfrak{y}} |\mathcal{E}(\mathfrak{o})|^{\mathfrak{q}} \, d\mathfrak{o}\right]^{\frac{1}{\mathfrak{q}}} \tag{3}$$

with equality if and only if $\Psi$ and $\mathcal{E}$ are proportional.

It is widely recognized that interval and fuzzy analysis provide essential tools for managing uncertain data. Interval analysis is commonly applied to problems where data contain inaccuracies, often due to measurement errors of various types. On the other hand, fuzzy analysis is suitable for addressing models created in situations where complete information about the problem is unavailable. Several generalized versions of Jensen's inequality in the context of integrals are available in the literature (see, for instance, [16]). These variations are primarily differentiated by the types of integrals applied. For example, the Sugeno integral is used in [17-20], the pseudo-integral is adopted in [21], while the Choquet integral is utilized in [22,23]. Costa and Román-Flores [24] introduced new fuzzy variants of Minkowski and Beckenbach integral inequalities, without utilizing the Sugeno integral. These innovative inequalities extend previously published interval versions of the Minkowski and Beckenbach inequalities by incorporating the concept of integrability for fuzzy-interval-valued functions through the Kaleva integral and a fuzzy order relation. The fuzzy order relation is defined level-wise using the Kulisch–Miranker order relation on the interval space. Additionally, the paper provides numerical examples to illustrate the practical application of the developed theory.

Khan *et al.,* [25, 26] introduced new types of fuzzy integral inequalities based on fuzzy fractional integrals, highlighting a relationship between inclusion relations and up-and-down fuzzy relations. Several illustrative examples are also presented to support the accuracy of the results. For more in-depth information, refer to the original study. For additional insights on fuzzy theory, consult [27-36] and the referenced works. Khastan and Rodríguez-López [37] recently introduced real-valued functions, utilizing Lebesgue measures to explore various properties of these functions within fuzzy contexts. Later, Khan and Guirao [38] extended this type of integral to encompass fractional integrals, specifically Riemann-Liouville fractional-like integrals over fuzzy domains. Additionally, they examined the properties of convex-like functions in fuzzy settings such that

If $\Psi: [\tilde{\mathrm{A}}] \to \mathfrak{N}$ is a convex-like function defined on the fuzzy number $[\tilde{\mathrm{A}}]^{\iota}$ for all $\iota \in [0,1]$, and $\theta, \lambda \in [\tilde{\mathrm{A}}]^{\iota}$ with $\theta < \lambda$, then

$$\Psi(\kappa\theta + (1-\kappa)\lambda) \preccurlyeq \kappa\Psi(\theta) + (1-\kappa)\Psi(\lambda) \tag{4}$$

where $\kappa \in [0,1]$. For basic concepts related to fuzzy sets see [39-41] and the references therein.

Inspired by the ongoing research work, this study introducess the triangular Hölder's like integral inequality over triangular fuzzy domain and its enhanced version to establish Hölder's power-mean-like integral inequality over fuzzy domain. Additionally, it introduces several classical and novel integral inequalities are also obtained as exceptional cases of triangular Hölder's like integral integral inequality. With the help of triangular Hölder's like integral integral inequality, triangular Minkowski's-like inequality and triangular Beckenbach's-like inequality over fuzzy domain. The paper also presents a few applications for unique representations of real-valued functions over triangular fuzzy numbers. Finally, some additional conclusions and future planes are discussed.

## 2. Preliminaries

Firstly, we offer the ideas and concepts needed for the follow-up. From Section 3, we offer the primary findings of the paper to guarantee its completion. We begin by defining a fuzzy set in such a way that:

Definition 1 [39]: A fuzzy subset $T$ of $\mathbb{R}$ is characterized by a mapping $\tilde{\mathrm{A}}: \mathbb{R} \to [0,1]$ known as the membership mapping of $T$, denoted as $\tilde{\mathrm{A}}: \mathbb{R} \to [0,1]$. Hence, for further investigation, we adopt this notation. We designate $\mathbb{C}$ to represent the set of all fuzzy subsets of $\mathbb{R}$.

In [40], Goetschel and Voxman introduced the main idea of fuzzy numbers as follows:

Let $\tilde{\mathrm{A}} \in \mathbb{C}$. Then, $\tilde{\mathrm{A}}$ is recognized as a fuzzy number or fuzzy interval if it satisfies the following properties:

(1) $\tilde{\mathrm{A}}$ should be normal if there exists $ɷ \in \mathbb{R}$ and $\tilde{\mathrm{A}}(ɷ) = 1$;

(2) $\tilde{\mathrm{A}}$ should be upper semi-continuous on $\mathbb{R}$ if for given $ɷ \in \mathbb{R}$, and $\varepsilon > 0$ there exist $\delta > 0$ such that $\tilde{\mathrm{A}}(ɷ) - \tilde{\mathrm{A}}(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|ɷ - y| < \delta$;

(3)$\tilde{\mathrm{A}}$ should be fuzzy convex, meaning $\tilde{\mathrm{A}}\big((1-\eta)ɷ + \eta y\big) \geq \min\big(\tilde{\mathrm{A}}(ɷ), \tilde{\mathrm{A}}(y)\big)$, for all $ɷ, y \in \mathbb{R}$, and $\eta \in [0,1]$;

(4) $\tilde{\mathrm{A}}$ should be compactly supported, i.e., $\mathrm{cl}\{ɷ \in \mathbb{R} |\ \tilde{\mathrm{A}}(ɷ) > 0\}$ is compact.

We designate $\mathbb{C}_0$ to represent the set of all fuzzy numbers of $\mathbb{R}$.

Definition 2 [39]: Given $\tilde{\mathrm{A}} \in \mathbb{C}_0$, the level sets or cut sets are defined as $[\tilde{\mathrm{A}}]^{\iota} = \{ɷ \in \mathbb{R} |\ \tilde{\mathrm{A}}(ɷ) > \iota\}$ for all $\iota \in [0,1]$.

From these definitions, we have

$$[\tilde{\mathrm{A}}]^{\iota} = [ʌ(\iota), ʋ(\iota)] \tag{5}$$

where

$ʌ(\iota) = \inf\{ɷ \in \mathbb{R} | \tilde{\mathrm{A}}(ɷ) \geq \iota\}$,

$ʋ(\iota) = \sup\{ɷ \in \mathbb{R} | \tilde{\mathrm{A}}(ɷ) \geq \iota\}$.

Remark 1 [40]: For each interval $[\mathfrak{v}, \mathfrak{y}] \in \mathcal{X}_C$, there characteristic function $\widetilde{[\mathfrak{v}, \mathfrak{y}]}: \mathbb{R} \to [0,1]$ defined by

$$\widetilde{[\mathfrak{v}, \mathfrak{y}]}(ɷ) = \begin{cases} 1 & ɷ \in [\mathfrak{v}, \mathfrak{y}] \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

So, in a way, we can consider that fuzzy numbers extend the set of closed intervals of real numbers, i.e., $\mathcal{X}_C \subseteq \mathbb{C}_0$, and consequently $\mathbb{R} \subseteq \mathbb{C}_0$ as well, since degenerated intervals can be interpreted as real numbers. Instead of representing $\widetilde{[\mathfrak{y}, \mathfrak{y}]}$, we simply use $\tilde{\mathfrak{y}}$. A fuzzy number $\tilde{\mathfrak{y}}$ is referred to as a crisp number or fuzzy singleton, as discussed in [40].

Recalling the concepts commonly found in the literature, if $\tilde{A}, \tilde{O} \in \mathbb{C}_0$ and $\iota \in \mathbb{R}$, then, for every $\iota \in [0, 1]$, the arithmetic operations are defined as follows:

$$[\tilde{A} \oplus \tilde{O}]^{\iota} = [\tilde{A}]^{\iota} + [\tilde{O}]^{\iota} \tag{7}$$

$$[\tilde{A} \otimes \tilde{O}]^{\iota} = [\tilde{A}]^{\iota} \times [\tilde{O}]^{\iota} \tag{8}$$

$$[\eta \odot \tilde{A}]^{\iota} = \eta . [\tilde{A}]^{\iota} \tag{9}$$

Theorem 2 [39]: The space $\mathbb{C}_0$ dealing with a supremum metric, i.e., for $\tilde{A}, \tilde{O} \in \mathbb{C}_0$

$$d_{\infty}(\tilde{A}, \tilde{O}) = \sup_{0 \leq \iota \leq 1} d_H([\tilde{A}]^{\iota}, [\tilde{O}]^{\iota}) \tag{10}$$

is a complete metric space, where $H$ denotes the well-known Hausdorff metric on space of intervals.

Now we recall some the concept of integral over fuzzy domain, where the integrable mappings are real-valued mappings over fuzzy domain.

Definition 3 [37]: If $\tilde{A} \in \mathbb{C}_0$, and $\Psi\colon [\tilde{A}]^{\iota} \subseteq \mathbb{R}^n \to \mathbb{R}$ is measurable on $[\tilde{A}]^0$ (and hence on every $[\tilde{A}]^{\iota}$, for all $\iota \in [0,1]$), then we define

$$\left(\int_{\tilde{A}} \Psi\right)(\iota) = \int_{[\tilde{A}]^{\iota}} \Psi(\text{ɷ})\, d\text{ɷ} \tag{11}$$

where the integral on the right-hand side is computed according to Lebesgue integration. We denote that $\Psi$ is integrable over the fuzzy domain if the integral $\int_{[\tilde{A}]^0} \Psi(\text{ɷ})\, d\text{ɷ}$ is finite. In such instances, the mapping is defined as:

$$\int_{\tilde{A}} \Psi : [0, 1] \to \mathbb{R}$$

$$\iota \to \left(\int_{\tilde{A}} \Psi\right)(\iota) = \int_{[\tilde{A}]^{\iota}} \Psi(\text{ɷ})\, d\text{ɷ} \tag{12}$$

Remark 2: By employing Remark 1, we derive the traditional definition of the integral, applicable to real-valued functions that are integrable.

## 3. Hölder-Like inequalities over triangular fuzzy $\mathbb{L}_{\boldsymbol{P}}$ space

In this section, we start with the new version of the following **triangular fuzzy $\mathbb{L}_{\boldsymbol{P}}$ space** such that

Considering the triangular fuzzy numbers ($T{\cdot}F{\cdot}N$s) $\tilde{A} = (\omega;\ \sigma, \xi)$, wherw $\omega \in \mathbb{R}$, and $\sigma, \xi \in \mathbb{R}$, thus

$$\tilde{A}(\text{ɷ}) = \begin{cases} \frac{\text{ʎ} - \omega + \sigma}{\sigma}, & \text{ʎ} \in [\omega - \sigma, \omega] \\ \frac{\omega + \xi - \text{ʎ}}{\xi}, & \text{ʎ} \in (\omega, \omega + \xi] \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Following is the geometric representation of $T{\cdot}F{\cdot}N$s (Figure 1):

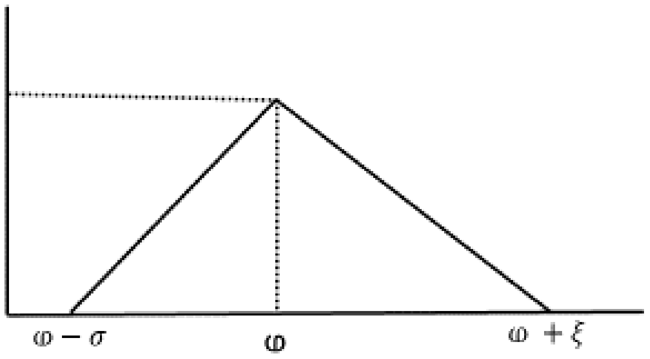

**Fig. 1.** Trapezoidal fuzzy number

whose parametrized form is $[\tilde{\mathrm{A}}]^{\iota} = [\omega - \sigma(1-\iota), \omega + \xi(1-\iota)]$, for all $\iota \in [0,1]$. Then,

$\mathbb{L}_P[\tilde{\mathrm{A}}] = \Big\{\varPsi | \varPsi : [\omega - \sigma(1-\iota), \omega + \xi(1-\iota)] \to \mathbb{R}$ is measurable on$[\omega - \sigma(1-\iota), \omega + \xi(1-\iota)]$ and $\int_{[\omega-\sigma(1-\iota), \omega+\xi(1-\iota)]} \varPsi(ɷ)\, dɷ < \infty$, for all $\iota \in [0,1]\Big\}$.

A measurable mapping defined on $\tilde{\mathrm{A}}$ is said to be $\mathfrak{p}^{\text{th}}$ power sum able, where $\mathfrak{p} \geq 1$, if

$$\iota \to \left(\int_{\tilde{\mathrm{A}}} \varPsi\right)(\iota) = \int_{[\tilde{\mathrm{A}}]^{\iota}} \varPsi(ɷ)\, dɷ < \infty \tag{14}$$

then, $\mathbb{L}_P$ space is denoted and defined as

$\mathbb{L}_{\mathfrak{p}}[\tilde{\mathrm{A}}] = \Big\{\varPsi | \varPsi : \tilde{\mathrm{A}} \to \mathbb{R}$ is measurable on$[\tilde{\mathrm{A}}]^{\iota}$and $\int_{[\tilde{\mathrm{A}}]^{\iota}} \varPsi(ɷ)\, dɷ < \infty\Big\}$.

*Remark 3:* Utilizing Remark 1 and Remark 2, we derive the classical $\mathbb{L}_{\mathfrak{p}}\widetilde{[\mathfrak{v}, \mathfrak{y}]}$ space.

*Theorem 3.* **(**Hölder like inequality) Suppose $\mathfrak{p} > 1$ and $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. If $\varPsi$ and $\mathcal{E}$ are two real functions defined on $[\tilde{\mathrm{A}}]^{\iota} = [\omega - \sigma(1-\iota), \omega + \xi(1-\iota)]$ such that $|\varPsi|^{\mathfrak{p}}$ and $|\mathcal{E}|^{\mathfrak{q}}$ are integrable functions on $[\tilde{\mathrm{A}}]^{\iota} = [\omega - \sigma(1-\iota), \omega + \xi(1-\iota)]$, then, for each $\iota \in [0,1]$

$$\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\varPsi(ɷ)\mathcal{E}(ɷ)|\, dɷ \leq \left[\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\varPsi(ɷ)|^{\mathfrak{p}}\, dɷ\right]^{\frac{1}{\mathfrak{p}}} \left[\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\mathcal{E}(ɷ)|^{\mathfrak{q}}\, dɷ\right]^{\frac{1}{\mathfrak{q}}} \tag{15}$$

with equality if and only if $\varPsi$ and $\mathcal{E}$ are proportional.

*Proof*. Since $[\tilde{\mathrm{A}}]^0 = [\omega - \sigma, \omega + \xi]$ (and hence on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$)

$\int_{[\tilde{\mathrm{A}}]^0} |\varPsi(ɷ)\mathcal{E}(ɷ)|\, dɷ = \int_{\omega-\sigma}^{\omega+\xi} |\varPsi(ɷ)\mathcal{E}(ɷ)|\, dɷ$, on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$.

Note that, If $\eta = \left[\int_{\omega-\sigma}^{\omega+\xi} |\varPsi(ɷ)|^{\mathfrak{p}}\, dɷ\right]^{\frac{1}{\mathfrak{q}}} = 0$, and $\xi = \left[\int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(ɷ)|^{\mathfrak{q}}\, dɷ\right]^{\frac{1}{\mathfrak{q}}} = 0$, it is obvious that equality will holds because functions $\varPsi$ and $\mathcal{E}$ are measurable on $[\tilde{\mathrm{A}}]^{\iota} = [\omega - \sigma(1-\iota), \omega + \xi(1-\iota)]$.

Considering $\eta = \left[\int_{\omega-\sigma}^{\omega+\xi} |\varPsi(ɷ)|^{\mathfrak{p}}\, dɷ\right]^{\frac{1}{\mathfrak{q}}} \neq 0$, and $\xi = \left[\int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(ɷ)|^{\mathfrak{q}}\, dɷ\right]^{\frac{1}{\mathfrak{q}}} \neq 0$ (and hence on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$).

*Case I.* Considering $\mathcal{u} = \frac{|\varPsi(ɷ)|}{\eta}, \mathcal{v} = \frac{|\mathcal{E}(ɷ)|}{\xi}$, Then, by using Auxiliary inequality, we have

$$\frac{|\varPsi(ɷ)||\mathcal{E}(ɷ)|}{\eta\xi} \leq \frac{|\varPsi(ɷ)|^{\mathfrak{p}}}{\mathfrak{p}\eta^{\mathfrak{p}}} + \frac{|\mathcal{E}(ɷ)|^{\mathfrak{q}}}{\mathfrak{q}\xi^{\mathfrak{q}}}.$$

Considering integration over $[\tilde{\mathrm{A}}]^0 = [\omega - \sigma, \omega + \xi]$ (and hence on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$) with respect to $ɷ$, we have

$$\frac{1}{\eta\xi}\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɷ)\mathcal{E}(ɷ)|\,dɷ \le \frac{1}{\mathfrak{p}\eta^{\mathfrak{p}}}\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɷ)|^{\mathfrak{p}}\,dɷ + \frac{1}{\mathfrak{q}\xi^{\mathfrak{q}}}\int_{\varpi-\sigma}^{\varpi+\xi}|\mathcal{E}(ɷ)|^{\mathfrak{q}}\,dɷ,$$

(and hence on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$), which implies that

$$\frac{1}{\eta\xi}\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɷ)\mathcal{E}(ɷ)|\,dɷ \le \frac{1}{\mathfrak{p}\eta^{\mathfrak{p}}}(\eta^{\mathfrak{p}}) + \frac{1}{\mathfrak{q}\xi^{\mathfrak{q}}}(\xi^{\mathfrak{q}}),$$

$= \frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$, (and hence on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$).

Then,

$$\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɷ)\mathcal{E}(ɷ)|\,dɷ \le \left[\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɷ)|^{\mathfrak{p}}\,dɷ\right]^{\frac{1}{\mathfrak{p}}}\left[\int_{\varpi-\sigma}^{\varpi+\xi}|\mathcal{E}(ɷ)|^{\mathfrak{q}}\,dɷ\right]^{\frac{1}{\mathfrak{q}}},$$

(and hence on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$),
which implies that

$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)\mathcal{E}(ɷ)|\,dɷ \le \left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)|^{\mathfrak{p}}\,dɷ\right]^{\frac{1}{\mathfrak{p}}}\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\mathcal{E}(ɷ)|^{\mathfrak{q}}\,dɷ\right]^{\frac{1}{\mathfrak{q}}},$$

for each $\iota \in [0,1]$.

*Remark 5:* If $\tilde{\mathrm{A}} = \widetilde{[\mathfrak{v},\mathfrak{y}]}$, then from (15), we get classical Hölder's-like inequality (3) for real-valued mappings.

*a) Applications*

When we obtain $|\Psi||\mathcal{E}| = \left(|\Psi|^{\frac{1}{\mathfrak{p}}}\right)\left(|\Psi|^{\frac{1}{\mathfrak{q}}}|\mathcal{E}|\right)$, as a straightforward outcome of the Hölder Inequality, we have the Hölder's power-mean-like integral inequality that follows:

*Theorem 4.* Suppose $\mathfrak{p} > 1$. If $\Psi$ and $\mathcal{E}$ are two real functions defined on fuzzy number $[\tilde{\mathrm{A}}]^{\iota} = [\varpi - \sigma(1-\iota), \varpi + \xi(1-\iota)]$ such that $|\Psi|$ and $|\Psi|^{\mathfrak{p}}|\mathcal{E}|$ are integrable functions on $[\tilde{\mathrm{A}}]^{\iota}$, then:

$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)\mathcal{E}(ɷ)|\,dɷ \le \left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)|^{\mathfrak{p}}\,dɷ\right)^{\frac{1}{\mathfrak{p}}}\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)||\mathcal{E}(ɷ)|^{\mathfrak{p}}\,dɷ\right)^{1-\frac{1}{\mathfrak{p}}} \tag{16}$$

*Proof.* By using same arguments like Theorem 3, it can be proved.

If $\mathfrak{p} = 2 = \mathfrak{q}$, then we attain the following outcome:

*Corollary 1:* (Cauchy-Schwarz's-like inequality) In accordance with the premises of Theorem 3, if $\mathfrak{p} = 2 = \mathfrak{q}$, then, it is evident that

$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)\mathcal{E}(ɷ)|\,dɷ \le \left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)|^{2}\,dɷ\right)^{\frac{1}{2}}\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\mathcal{E}(ɷ)|^{2}\,dɷ\right)^{\frac{1}{2}}, \tag{17}$$

for each $\iota \in [0,1]$.

## 4. Minkowski-Like inequality

*Theorem 5.* (Minkowski's-like inequality) Suppose $\mathfrak{p} \ge 1$. If $\Psi$ and $\mathcal{E}$ are two real functions defined on $[\tilde{\mathrm{A}}]^{\iota} = [\varpi - \sigma(1-\iota), \varpi + \xi(1-\iota)]$ such that $|\Psi|^{\mathfrak{p}}$ and $|\mathcal{E}|^{\mathfrak{p}}$ are integrable functions on $[\tilde{\mathrm{A}}]^{\iota}$, then, for each $\iota \in [0,1]$

$$\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)+\mathcal{E}(ɷ)|^{\mathfrak{p}}\,dɷ\right)^{\frac{1}{\mathfrak{p}}} \le \left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɷ)|^{\mathfrak{p}}\,dɷ\right]^{\frac{1}{\mathfrak{p}}} + \left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\mathcal{E}(ɷ)|^{\mathfrak{p}}\,dɷ\right]^{\frac{1}{\mathfrak{p}}}. \tag{18}$$

with equality if and only if $\Psi$ and $\mathcal{E}$ are proportional.

If $1 > \mathfrak{p} > 0$, then

$$\left(\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right)^{\frac{1}{\mathfrak{p}}} \geq \left[\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\Psi(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}} + \left[\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}}. \qquad (19)$$

*Proof. Case I.* Suppose that $\mathfrak{p} = 1$ and we know that

$|\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})| \leq |\Psi(\text{ɷ})| + |\mathcal{E}(\text{ɷ})|$.

Considering integration on both side over $[\tilde{\mathrm{A}}]^0 = [\omega - \sigma, \omega + \xi]$ (and hence on every $[\tilde{\mathrm{A}}]^{\iota}$, for all $\iota \in [0,1]$), we have

$$\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|\, d\text{ɷ} \leq \int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ})|\, d\text{ɷ} + \int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(\text{ɷ})|\, d\text{ɷ},$$

which implies that

$$\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|\, d\text{ɷ} \leq \int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\Psi(\text{ɷ})|\, d\text{ɷ} + \int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)} |\mathcal{E}(\text{ɷ})|\, d\text{ɷ}.$$

*Case II.* Consider that $\mathfrak{p} > 1$ and that $\mathfrak{p}$ and $\mathfrak{q}$ are conjugate indices. Then,

$$\begin{aligned}\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ} &= \int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|\, |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}-1}\, d\text{ɷ} \\ &= \int_{\omega-\sigma}^{\omega+\xi} (|\Psi(\text{ɷ})| + |\mathcal{E}(\text{ɷ})|)|\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}-1}\, d\text{ɷ} \\ &= \int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ})||\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}-1}\, d\text{ɷ} + \int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(\text{ɷ})||\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}-1}\, d\text{ɷ},\end{aligned}$$

and hence on every $[\tilde{\mathrm{A}}]^{\iota} = [\omega - \sigma(1-\iota), \omega + \xi(1-\iota)]$, for all $\iota \in [0,1]$.

By using Hölder like Inequality, we have

$$\begin{aligned}\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ} &\leq \left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}} \left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{(\mathfrak{p}-1)\mathfrak{q}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{q}}} \\ &\quad + \left[\int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}} \left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{(\mathfrak{p}-1)\mathfrak{q}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{q}}} \\ &= \left(\left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}} + \left[\int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}}\right) \left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{(\mathfrak{p}-1)\mathfrak{q}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{q}}},\end{aligned}$$

which implies, we have

$$\begin{aligned}&\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ} \\ &\leq \left(\left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}} + \left[\int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}}\right) \left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{q}}} \\ &\leq \left(\left[\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}} + \left[\int_{\omega-\sigma}^{\omega+\xi} |\mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right]^{\frac{1}{\mathfrak{p}}}\right) \left[\left(\int_{\omega-\sigma}^{\omega+\xi} |\Psi(\text{ɷ}) + \mathcal{E}(\text{ɷ})|^{\mathfrak{p}}\, d\text{ɷ}\right)^{\frac{1}{\mathfrak{p}}}\right]^{\frac{\mathfrak{p}}{\mathfrak{q}}}.\end{aligned}$$

From above inequality, we have

$$\left(\left(\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɵ)+\mathcal{E}(ɵ)|^{\mathfrak{p}}\,dɵ\right)^{\frac{1}{\mathfrak{p}}}\right)^{\mathfrak{p}}\left[\left(\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɵ)+\mathcal{E}(ɵ)|^{\mathfrak{p}}\,dɵ\right)^{\frac{1}{\mathfrak{p}}}\right]^{-\frac{\mathfrak{p}}{\mathfrak{q}}}$$
$$\leq\left[\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɵ)|^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}}+\left[\int_{\varpi-\sigma}^{\varpi+\xi}|\mathcal{E}(ɵ)|^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}},$$

implies that

$$\left(\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɵ)+\mathcal{E}(ɵ)|^{\mathfrak{p}}\,dɵ\right)^{\frac{1}{\mathfrak{p}}}\leq\left[\int_{\varpi-\sigma}^{\varpi+\xi}|\Psi(ɵ)|^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}}+\left[\int_{\varpi-\sigma}^{\varpi+\xi}|\mathcal{E}(ɵ)|^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}},$$

and hence on every $[\tilde{\mathrm{A}}]^{\iota}=[\varpi-\sigma(1-\iota),\varpi+\xi(1-\iota)]$, for all $\iota\in[0,1]$.
Hence,

$$\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɵ)+\mathcal{E}(ɵ)|^{\mathfrak{p}}\,dɵ\right)^{\frac{1}{\mathfrak{p}}}\leq\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\Psi(ɵ)|^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}}+\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}|\mathcal{E}(ɵ)|^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}}.$$

*Remark 6.* If $\tilde{\mathrm{A}}=\widetilde{[\mathfrak{v},\mathfrak{y}]}$, then from (18) and (19), we get classical Minkowski's inequality for real-valued mappings.

*Example 1.* Consider the trapezoidal fuzzy numbers $\tilde{\mathrm{A}}=(2;2,2)$, that is

$$\tilde{\mathrm{A}}(ɵ)=\begin{cases}1, & ʎ=2\\ \frac{ʎ-\frac{1}{2}}{2}, & ʎ\in[0,2]\\ \frac{4-ʎ}{2}, & ʎ\in[2,2+2]\\ 0, & \text{otherwise,}\end{cases}\tag{20}$$

whose parametrized form is $[\tilde{\mathrm{A}}]^{\iota}=[2+2(\iota-1),2+2(1-\iota)]$, for all $\iota\in[0,1]$. Let $\mathfrak{p}=\frac{1}{2}$, and $\Psi(ɵ)=ɵ$ and $\mathcal{E}(ɵ)=ɵ^2$ be the real-valued mappings on fuzzy domain $\tilde{\mathrm{A}}$.

$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\Psi(ɵ)\big)^{\mathfrak{p}+1}\,dɵ=\frac{8\sqrt{2}}{5}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)$$
$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\Psi(ɵ)\big)^{\mathfrak{p}}\,dɵ=\frac{4\sqrt{2}}{3}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)$$
$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}+1}\,dɵ=32(2-4\iota+3\iota^2-\iota^3)$$
$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}}\,dɵ=8-8\iota$$

$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\Psi(ɵ)+\mathcal{E}(ɵ)\big)^{\mathfrak{p}+1}\,dɵ=\frac{16}{25}\sqrt{2}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)\left((5-2\iota)^{\frac{5}{2}}-(1+2\iota)^{\frac{5}{2}}\right)$$
$$\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\Psi(ɵ)+\mathcal{E}(ɵ)\big)^{\mathfrak{p}}\,dɵ=\frac{8}{9}\sqrt{2}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)\left((5-2\iota)^{\frac{3}{2}}-(1+2\iota)^{\frac{3}{2}}\right).$$

Now

$$\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\Psi(ɵ)\big)^{\mathfrak{p}+1}\,dɵ\right]^{\frac{1}{\mathfrak{p}+1}}=\left(\frac{8\sqrt{2}}{5}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)\right)^{\frac{2}{3}}$$
$$\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}+1}\,dɵ\right]^{\frac{1}{\mathfrak{p}+1}}=\big(32(2-4\iota+3\iota^2-\iota^3)\big)^{\frac{2}{3}}$$

$$\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\Psi(\theta)+\mathcal{E}(\theta)\right)^{\mathfrak{p}+1}d\theta\right)^{\frac{1}{\mathfrak{p}+1}}=\left[\frac{16}{25}\sqrt{2}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)\left((5-2\iota)^{\frac{5}{2}}-(1+2\iota)^{\frac{5}{2}}\right)\right]^{\frac{2}{3}}$$

Then

$$\left[\frac{16}{25}\sqrt{2}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)\left((5-2\iota)^{\frac{5}{2}}-(1+2\iota)^{\frac{5}{2}}\right)\right]^{\frac{2}{3}}$$
$$\leq\left(\frac{8\sqrt{2}}{5}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)\right)^{\frac{2}{3}}+\left(32(2-4\iota+3\iota^2-\iota^3)\right)^{\frac{2}{3}},$$

for each $\iota\in[0,1]$.

Hence, Minkowski-like inequality (18) is verified. For (19), we have

$$\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\Psi(\theta)\right)^{\mathfrak{p}}d\theta\right]^{\frac{1}{\mathfrak{p}}}=\left(\frac{4\sqrt{2}}{3}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)\right)^{2},$$

$$\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\mathcal{E}(\theta)\right)^{\mathfrak{p}}d\theta\right]^{\frac{1}{\mathfrak{p}}}=(8-8\iota)^2,$$

$$\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\Psi(\theta)+\mathcal{E}(\theta)\right)^{\mathfrak{p}}d\theta\right)^{\frac{1}{\mathfrak{p}}}=\left[\frac{8}{9}\sqrt{2}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)\left((5-2\iota)^{\frac{3}{2}}-(1+2\iota)^{\frac{3}{2}}\right)\right]^{2}.$$

Then,

$$\left[\frac{8}{9}\sqrt{2}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)\left((5-2\iota)^{\frac{3}{2}}-(1+2\iota)^{\frac{3}{2}}\right)\right]^{2}\geq\left(\frac{4\sqrt{2}}{3}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)\right)^{2}+(8-8\iota)^2,$$

for each $\iota\in[0,1]$.

## 5. Beckenbach's inequality

*Theorem 6.* (Beckenbach's inequality) Suppose $1>\mathfrak{p}>0$. If $\Psi$ and $\mathcal{E}$ are two real functions defined on $[\tilde{\mathrm{A}}]^{\iota}=[\varpi-\sigma(1-\iota),\varpi+\xi(1-\iota)]$ and $\Psi(\theta)>0,\mathcal{E}(\theta)>0$, then

$$\frac{\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\Psi(\theta)+\mathcal{E}(\theta)\right)^{\mathfrak{p}+1}d\theta}{\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\Psi(\theta)+\mathcal{E}(\theta)\right)^{\mathfrak{p}}d\theta}\leq\frac{\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\Psi(\theta)\right)^{\mathfrak{p}+1}d\theta}{\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\Psi(\theta)\right)^{\mathfrak{p}}d\theta}+\frac{\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\mathcal{E}(\theta)\right)^{\mathfrak{p}+1}d\theta}{\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\left(\mathcal{E}(\theta)\right)^{\mathfrak{p}}d\theta}. \tag{21}$$

with equality if $\Psi$ and $\mathcal{E}$ are proportional.

*Proof.* Considering

$$l_1=\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}(\Psi)^{\mathfrak{p}+1}d\theta\right)^{\frac{1}{\mathfrak{p}+1}},\qquad l_2=\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}(\mathcal{E})^{\mathfrak{p}+1}d\theta\right)^{\frac{1}{\mathfrak{p}+1}}$$

and

$$J_1=\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}(\Psi)^{\mathfrak{p}}d\theta\right)^{\frac{1}{\mathfrak{p}}},\qquad J_2=\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}(\mathcal{E})^{\mathfrak{p}}d\theta\right)^{\frac{1}{\mathfrak{p}}}.$$

Now by using Randon inequality for real number, we have

$$\frac{{l_1}^{\mathfrak{p}+1}}{{J_1}^{\mathfrak{p}}}+\frac{{l_2}^{\mathfrak{p}+1}}{{J_2}^{\mathfrak{p}}}\geq\frac{(l_1+l_2)^{\mathfrak{p}+1}}{(J_1+J_2)^{\mathfrak{p}}},$$

that is to say

$$\frac{\int_{\varpi-\sigma}^{\varpi+\xi}\left(\Psi(\theta)\right)^{\mathfrak{p}+1}d\theta}{\int_{\varpi-\sigma}^{\varpi+\xi}\left(\Psi(\theta)\right)^{\mathfrak{p}}d\theta}+\frac{\int_{\varpi-\sigma}^{\varpi+\xi}\left(\mathcal{E}(\theta)\right)^{\mathfrak{p}+1}d\theta}{\int_{\varpi-\sigma}^{\varpi+\xi}\left(\mathcal{E}(\theta)\right)^{\mathfrak{p}}d\theta}\geq\frac{\left(\left(\int_{\varpi-\sigma}^{\varpi+\xi}\left(\Psi(\theta)\right)^{\mathfrak{p}+1}d\theta\right)^{\frac{1}{\mathfrak{p}+1}}+\left(\int_{\varpi-\sigma}^{\varpi+\xi}\left(\mathcal{E}(\theta)\right)^{\mathfrak{p}+1}d\theta\right)^{\frac{1}{\mathfrak{p}+1}}\right)^{\mathfrak{p}+1}}{\left(\left(\int_{\varpi-\sigma}^{\varpi+\xi}\left(\Psi(\theta)\right)^{\mathfrak{p}}d\theta\right)^{\frac{1}{\mathfrak{p}}}+\left(\int_{\varpi-\sigma}^{\varpi+\xi}\left(\mathcal{E}(\theta)\right)^{\mathfrak{p}}d\theta\right)^{\frac{1}{\mathfrak{p}}}\right)^{\mathfrak{p}}} \tag{22}$$

and hence on every $[\tilde{A}]^{\iota} = [\varpi - \sigma(1-\iota), \varpi + \xi(1-\iota)]$, for all $\iota \in [0,1]$.

Now because $1 > \mathfrak{p} > 0$, then $2 > \mathfrak{p}+1 > 1$, from (21) and (22), we achieve

$$\left[\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)+\mathcal{E}(ɵ)\big)^{\mathfrak{p}+1}\,dɵ\right]^{\frac{1}{\mathfrak{p}+1}} \le \left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)\big)^{\mathfrak{p}+1}\,dɵ\right)^{\frac{1}{\mathfrak{p}+1}} + \left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}+1}\,dɵ\right)^{\frac{1}{\mathfrak{p}+1}} \tag{23}$$

and

$$\left[\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)+\mathcal{E}(ɵ)\big)^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}} \ge \left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)\big)^{\mathfrak{p}}\,dɵ\right)^{\frac{1}{\mathfrak{p}}} + \left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}}\,dɵ\right)^{\frac{1}{\mathfrak{p}}} \tag{24}$$

As we know that, if $a, b, c > 0$, then we have

$$a \ge c \Leftrightarrow \frac{a}{b} \ge \frac{c}{b}, \tag{25}$$

$$b \le c \Leftrightarrow \frac{a}{b} \ge \frac{a}{c} \tag{26}$$

Finally, from (23), (24), (25) and (26), we have

$$\frac{\left(\left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)\big)^{\mathfrak{p}+1}dɵ\right)^{\frac{1}{\mathfrak{p}+1}}+\left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}+1}dɵ\right)^{\frac{1}{\mathfrak{p}+1}}\right)^{\mathfrak{p}+1}}{\left(\left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)\big)^{\mathfrak{p}}dɵ\right)^{\frac{1}{\mathfrak{p}}}+\left(\int_{\varpi-\sigma}^{\varpi+\xi}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}}dɵ\right)^{\frac{1}{\mathfrak{p}}}\right)^{\mathfrak{p}}} \ge \frac{\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)+\mathcal{E}(ɵ)\big)^{\mathfrak{p}+1}dɵ}{\int_{\varpi-\sigma}^{\varpi+\xi}\big(\Psi(ɵ)+\mathcal{E}(ɵ)\big)^{\mathfrak{p}}dɵ} \tag{27}$$

and hence on every $[\tilde{A}]^{\iota} = [\varpi - \sigma(1-\iota), \varpi + \xi(1-\iota)]$, for all $\iota \in [0,1]$. Hence, from (26) and (27), we conclude the required result.

*Remark 7.* If $\tilde{A} = \widetilde{[\mathfrak{v}, \mathfrak{y}]}$, then from (21), we get following classical Beckenbach's inequality for real-valued mappings.

*Example 2.* Consider the trapezoidal fuzzy numbers $\tilde{A} = (2; 2,2)$, with parametrized form $[\tilde{A}]^{\iota} = [2 + 2(\iota - 1), 2 + 2(1-\iota)]$, for all $\iota \in [0,1]$, taken from Example 1. Let $\mathfrak{p} = \frac{1}{2}$, and $\Psi(ɵ) = ɵ$ and $\mathcal{E}(ɵ) = ɵ^2$ be the real-valued mappings on fuzzy domain $\tilde{A}$.

For (20), we have

$$\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\Psi(ɵ)\big)^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}} = \left(\frac{4\sqrt{2}}{3}\left((2-\iota)^{\frac{3}{2}} - \iota^{\frac{3}{2}}\right)\right)^2,$$

$$\left[\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\mathcal{E}(ɵ)\big)^{\mathfrak{p}}\,dɵ\right]^{\frac{1}{\mathfrak{p}}} = (8-8\iota)^2,$$

$$\left(\int_{\varpi-\sigma(1-\iota)}^{\varpi+\xi(1-\iota)}\big(\Psi(ɵ)+\mathcal{E}(ɵ)\big)^{\mathfrak{p}}\,dɵ\right)^{\frac{1}{\mathfrak{p}}} = \left[\frac{8}{9}\sqrt{2}\left((2-\iota)^{\frac{3}{2}} - \iota^{\frac{3}{2}}\right)\left((5-2\iota)^{\frac{3}{2}} - (1+2\iota)^{\frac{3}{2}}\right)\right]^2.$$

Then,

$$\left[\frac{8}{9}\sqrt{2}\left((2-\iota)^{\frac{3}{2}} - \iota^{\frac{3}{2}}\right)\left((5-2\iota)^{\frac{3}{2}} - (1+2\iota)^{\frac{3}{2}}\right)\right]^2 \ge \left(\frac{4\sqrt{2}}{3}\left((2-\iota)^{\frac{3}{2}} - \iota^{\frac{3}{2}}\right)\right)^2 + (8-8\iota)^2,$$

for each $\iota \in [0,1]$.

For, triangular Beckenbach inequality (21), we have

$$\frac{\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)}\left(\Psi(\vartheta)+\mathcal{E}(\vartheta)\right)^{\mathfrak{p}+1}d\vartheta}{\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)}\left(\Psi(\vartheta)+\mathcal{E}(\vartheta)\right)^{\mathfrak{p}}d\vartheta}=\frac{\frac{16}{25}\sqrt{2}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)\left((5-2\iota)^{\frac{5}{2}}-(1+2\iota)^{\frac{5}{2}}\right)}{\frac{8}{9}\sqrt{2}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)\left((5-2\iota)^{\frac{3}{2}}-(1+2\iota)^{\frac{3}{2}}\right)}$$

$$\frac{\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)}\left(\Psi(\vartheta)\right)^{\mathfrak{p}+1}d\vartheta}{\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)}\left(\Psi(\vartheta)\right)^{\mathfrak{p}}d\vartheta}=\frac{\frac{8\sqrt{2}}{5}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)}{\frac{4\sqrt{2}}{3}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)}$$

$$\frac{\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)}\left(\mathcal{E}(\vartheta)\right)^{\mathfrak{p}+1}d\vartheta}{\int_{\omega-\sigma(1-\iota)}^{\omega+\xi(1-\iota)}\left(\mathcal{E}(\vartheta)\right)^{\mathfrak{p}}d\vartheta}=\frac{32\left(2-4\iota+3\iota^{2}-\iota^{3}\right)}{8-8\iota}$$

From (25)

$$\frac{\frac{16}{25}\sqrt{2}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)\left((5-2\iota)^{\frac{5}{2}}-(1+2\iota)^{\frac{5}{2}}\right)}{\frac{8}{9}\sqrt{2}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)\left((5-2\iota)^{\frac{3}{2}}-(1+2\iota)^{\frac{3}{2}}\right)}\leq\frac{\frac{8\sqrt{2}}{5}\left((2-\iota)^{\frac{5}{2}}-\iota^{\frac{5}{2}}\right)}{\frac{4\sqrt{2}}{3}\left((2-\iota)^{\frac{3}{2}}-\iota^{\frac{3}{2}}\right)}+\frac{32\left(2-4\iota+3\iota^{2}-\iota^{3}\right)}{8-8\iota},$$

for each $\iota\in[0,1]$.

## 6. Conclusion

By utilizing the Lebesgue integral for real-valued functions over fuzzy domain as the corresponding expectation, we gain a reliable framework for managing and quantifying uncertainty. Building on this idea, we apply the Lebesgue integral to demonstrate triangular Hölder's-like inequality for real-valued functions over fuzzy domain. Furthermore, we explore additional power mean Hölder's-like inequality for real-valued functions over fuzzy domain to identify the appropriate form of triangular Hölder's-like inequality. In contrast, we examine the connections between triangular Hölder, and triangular Cauchy–Schwarz, triangular Minkowski's-like inequality, and triangular Beckenbach's-like inequalities domain within the fuzzy interval setting. This analysis contributes to the generalization of several classical integral inequalities in the real-valued context. To conclude, we present updated versions of Hölder and Cauchy–Schwarz inequalities using triangular fuzzy number, incorporating submodular measures and comonotone functions—topics we plan to explore further in future work. Additionally, examples and applications are provided to illustrate the results achieved.

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## Conflicts of Interest

The authors declare no conflicts of interest.

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