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Investigation of Threats to Data States in Data Loss Prevention Using the Concept of Complex Linear Diophantine Fuzzy Relations

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ABSTRACT

In today's digital age, many operational decisions in businesses rely on data sources, enabling organizations to enhance productivity, make informed decisions, and gain competitive advantages. However, businesses also face data breaches involving sensitive information—such as financial records, intellectual property, and customer personal data—which may be compromised inadvertently. These threats can often be mitigated by implementing robust cybersecurity measures, such as Data Loss Prevention (DLP), to ensure proper monitoring and control of all organizational data, enforce policies without exceptions, and prevent unauthorized data transfers or rule violations. Despite these measures, uncertainties remain regarding the efficacy of identifying threats at various stages of data loss to mitigate their adverse effects through effective cybersecurity. To address this, this paper introduces Complex Linear Diophantine Fuzzy Relations (CLDFRs). For the first time in fuzzy set theory, we analysed the relationships between various threats and components of DLP-based data loss solutions. Additionally, we present the concept of Hasse diagrams for Complex Linear Diophantine Fuzzy Sets and Relations to examine different cybersecurity methods and procedures. This approach helps determine the most effective strategy based on Hasse diagram analysis. Furthermore, after applying specific constraints to the decision-making process, the optimal cybersecurity approach is selected. Finally, a comparative analysis demonstrates the advantages of the proposed methods.

1. Introduction

Since many problems concerning with uncertainty and vagueness are unable to be handled efficiently by conventional set theory, which deals with crisp set and binary membership irrespective of whether an element is involved in a set or not. By integrating doubts and inaccuracies relevant to information based on real-world situations and human deductive reasoning, thus Zadeh [1] presented the concept of fuzzy set theory in 1965 that renewed conventional set theory. In order to indicate the extent of element's existence in a set, a partial membership is enabled by fuzzy set theory through which a gradual progression of a component's existence in unit interval is illustrated. The

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versatility of FS in expressing inaccuracy and unpredictability renders it a useful tool to characterize a complex system in which definite boundaries are difficult to identify or illusive. Furthermore, Klir [2] presented a definition for the idea of relation among crisp set through which only yes-and-no type problems can be solved. Since it deals with precise information so this theory of set fails to represent uncertainty. Thus, an idea of fuzzy relation (FRs), or relations for FSs was put forward by Mendal [3]. FRs can indicate the degree, effectiveness and extent of good relations among any two FSs instead of restricted to address only binary terms. A set of axioms for a comparatively basic form of FSs was provided by Goguen [4]. Zywica [5] utilized FSs to analyze uncertainties in medical. Uncertainty, gradualness and bipolarity via FRs were investigated by Dubis and Prade [6].

Furthermore, to address uncertainty and inaccuracy in complex structures, Ramot *et al.,* [7] put forward the concept of Complex fuzzy set (CFSs) in 2002, in which a complex number are utilized to assign to membership degrees with real and imaginary parts known as amplitude term and phase term respectively along with its values lies in unit interval, are presented. Moreover, the concept of complex fuzzy relations (CFRs) through which a relation between two CFs is determined, was presented by Ramot *et al.,* [7]. Yazdanbakhsh and Dick [8] investigated the CFs. Liu [9] introduced the use of distance measurement on CFs in decision-making applications. CFs was examined by Sobhi [10]. An application for CFs were put forward by Tamir *et al.,* [11].

In some cases, it become uncertain to identify an extent of set being a member or non-member, thus an intuitionistic fuzzy set (IFS) as a situation-handling broadening of FS was introduced by Atanassov [12] in 1999. The values of membership and non-membership degrees must vary in unit interval respectively. Furthermore, the summation of membership and non-membership does not surpass the unit interval. The IFS become FS if the non-membership degree equals to zero. Burillo *et al.,* [13] developed the invention of intuitionistic fuzzy relation (IFRs). Furthermore, Alkouri *et al.,* [14] put forward a novel proposal of complex intuitionistic fuzzy set (CIFS) in 2012, in order to deal with complex structure relating with membership and non-membership of an element in a set. The IFSs was further developed by Rehman *et al.,* [15]. From a dynamic viewpoint, IFSs was further reviewed by Yu *et al.,* [16]. Moreover, the field of medicine [17], aggregation operator [18], pattern recognition [19] and decision making [20] has given more attention to IFSs. Complex relations were used by Nasir *et al.,* [21], [22] to evaluate economic relationships. Some CIFS applications for Artificial Intelligence were put forward by Garg and Rani [23].

Moreover, under certain conditions, the limit which is permitted for the total of membership and non-membership degree exceeds boundary. Thus, a Pythagorean fuzzy set (PyFSs) was introduced by Yager [24], [25] assuring that totality of square of membership and non-membership does not surpass the unit interval. Furthermore, in order to deal with complex structures, Ullah *et al.,* [26] came up with a proposal of complex Pythagorean fuzzy set (CPyFSs). A PyFSs was examined by Peng *et al.,* [27]. Saikia *et al.,* [28] presented an application in transportation problem based on advanced similarity measure in PyFSs. Khan *et al.,* [29] expanded PyFSs. Pan *et al.,* [30] put forward the quaternion model of PyFSs. Labassi *et al.,* [31] introduced application in visualization technology via novel approach through CPyFSs. Akram *et al.,* [32] presented optimization technique through CPyFSs for making decision.

Sometimes, PyFSs fails as a totality of square goes beyond the limit, so thus Yager [33] in 2016 suggested an idea of q-rung orthopair fuzzy set (q-ROFS) with a q-rung variable ensuring that totality of q-exponent of membership and non-memberships respectively must be within unit interval. Furthermore, Liu *et al.,* [34] and Garg *et al.,* [35] presented the proposal of complex q-rung orthopair fuzzy set (Cq-ROFS) through which a complex structures facing uncertainties will be handled. The q-ROFS was further investigated by Peng and Luo [36]. The first method for quantifying knowledge related to q-ROFS was put forward by Khan *et al.,* [37]. Demir *et al.,* [38] presented an extensive approach for decision-making in service industry through q-ROFSs. Akram *et al.,* [39] introduced innovative approach to decision-making using Cq-ROFSs. Javeed *et al.,* [40] proposed some application of Cq-ROFSs in medical.

Furthermore, an investigation was conducted in order to concern some limitations related to membership and non-membership mapping in FSs, IFSs, PyFSs and q-ROFSs models. Thus, in 2019 Riaz and Hashmi [41] came up with an idea of linear Diophantine fuzzy set (LDFSs) in order to get around these restrictions by inserting reference parameters to IFS structure. By reducing the shortcomings of current approaches of other set, it was declared that LDFS structure enables unrestricted data determination in real-world situations. Additionally, LDFSs is demonstrated to be broader than FSs, IFSs, PyFSs and q-ROFSs through the arbitrary attribute of reference parameter. But in some cases, when it comes to resolve complex structure with C-valued membership degree, non-membership degrees and reference parameters respectively, Kamacı [42] came up with an extension of LDFSs as complex linear Diophantine fuzzy set (CLDFSs). CLDFSs appears with C-valued membership degree, non-membership degrees and reference parameters respectively. Ayub et al. [43], [44] put forward some linear Diophantine fuzzy relation (LDFRs) based novel approaches. Zia *et al.,* [45] presented applications of CLDFSs in multi-attribute decision-making. Guan *et al.,* [46] came up with an application of CLDFSs in engineering.

This paper presents the idea of Cartesian product among two CLDFSs. Additionally, through the idea of CP of two CLDFSs, advanced invention of a complex linear Diophantine fuzzy relation CLDFR is summarized. Furthermore, various kinds of CLDFRs such as complex linear Diophantine equivalence fuzzy relation, complex linear Diophantine partial order fuzzy relation, complex linear Diophantine composite fuzzy relation, complex linear Diophantine total order fuzzy relation, complex linear Diophantine converse fuzzy relation and much more along with suitable examples, have also been discussed. Besides these, for CLDFRs certain results have been established. Furthermore, for complex linear Diophantine partial order fuzzy sets and relation, a concept of Hasse diagram have also been shown. Also, following concepts such as maximum element, minimum element, maximal element, minimal element, supremum, infimum, upper and lower bounds have been mentioned. In order to compare a current concept presented in this paper to previous structures, it follows that CLDFSs and CLDFRs are dominant over FSs, CFSs, IFSs, CIFSs, CPyFSs, CqROFSs and LDFSs. As, the relationship between CLDFSs examined by CLDFRs, consist of complex-valued membership degree, non-membership degree and parameters. Furthermore, compared to other ideas, these conceptions are far more adept at handling uncertainty. They can handle data with multiple variables more precisely due to the presences of complex-valued membership degree, non-membership degree and parameters without any restriction to any limitation.

Nowadays, Businesses handled considerably more data as they started to digitize their operations. While there are many benefits associated with this digital transition, like increased efficiency and connection, there are also new threats. Businesses began to recognize that breaches, both intentional and unintentional, may occur to their sensitive and important data, which included financial data, intellectual property, and personal information about consumers and staff. As cybertheft often appeared in media headlines, individuals and institutions started advocating for more stringent data protection regulations. Around this same period, legal entities began to pay attention. For instance, the first law safeguarding the privacy of personal consumer information was approved in California in 2003. As more information is transmitted and stored digitally, the danger of data breaches grows. The risks were made clear by well-publicized data breaches that occurred in the late 20th and early 21st centuries. Thus, the development of cybersecurity as a whole is entwined with the historical context of DLP software. Solutions to counter threats also developed and got more complex over time. These applications might provide complete security throughout the data handling process by identifying, monitoring, and safeguarding data when it is in use, in motion, and at rest. Thus, businesses of all sizes utilize DLP software as the cornerstone of their information security procedures to protect their most precious assets. The development of DLP solutions from the early days of data breaches to the current state of sophistication illustrates the constant innovation and adaptation required to safeguard sensitive data in an increasingly digital society. In order to deal with the uncertainty of data loss resulted from threats, we implemented fuzzy theory to get over all of this vagueness. Thus, a relationship between components of data loss solution and threats in data loss preservation is numerically studied in this article. This studied include potency and impotency of data loss preventive measure against various sources. Additionally, the present paper also suggests a way to evaluate a various data loss state on which a different threat can effectively attack and among these, select the most appropriate one for DLP. The concept of Hasse diagram and complex linear Diophantine partial order fuzzy relation assist as a basis for this novel approach. Thus, other comparable approaches that are already in literature review are matched with suggested strategies. As fuzzy set theory has not yet recognized the complex relations in CLDFSs, thus a supremacy and authenticity are initiated by numeric problems. From now on, there is an effective possibility for prospective studies to be conducted in order to investigate these structures.

The continued sections of paper are ordered as follow:

The paper is based on certain established concept in fuzzy set theory, which are reconsidered in section 2. In section 3, novel innovations for CLDFRs are illustrated, such as relation between two CLDFSs resulting from CPs, CLDFRs types and proof of various theorems. Section 4 present a Hasse diagram along with some helpful definitions and properties to elaborate complex linear Diophantine partial order fuzzy set and relation. In section 5, the application of CLDFSs and CLDFRs is submitted. The effectiveness of various threats on any state of data loss in DLP is explored. Section 6 compares the suggested structures with the one that are currently in use in the field of fuzzy set theory. Lastly, conclusion finishes the paper.

2. Preliminaries

This section put forward some fundamental interpretations along with examples, comprising fuzzy set (FS), complex fuzzy set (CFS), intuitionistic fuzzy set (IFS), complex intuitionistic fuzzy set (CIFS), complex Pythagorean fuzzy set (CPyFS), complex q-rung orthopair fuzzy set (Cq-ROFS) and linear Diophantine fuzzy set (LDFS).

Definition 1. [1] A set τ_f on a universal set χ, known as fuzzy set is of the following form

$$\tau_f = \{(a, \mathfrak{M}(a)) : a \in \chi\}$$

Whereas a mapping $\mathfrak{M}: \chi \to [0,1]$ assigns a membership grade to each element of a set.

Example 1. The set $\tau_f = \{(a_\chi, 0.62), (b_\chi, 0), (c_\chi, 0.91), (d_\chi, 0.19), (e_\chi, 0.56)\}$ represents FS.

Definition 2. [7] A set τ_f on a universal set χ, known as complex fuzzy set is of the following form

$$\tau_f = \{(a, \mathfrak{M}_{\mathbb{C}}(a)) : a \in \chi\}$$

Whereas a mapping $\mathfrak{M}_{\mathbb{C}}: \chi \to \mathbb{Z} \ni 0 \leq |\mathbb{Z}| \leq 1$ assigns a membership grade to each element of a set and $\mathbb{Z}$ is a complex number.

Furthermore, the complex fuzzy set can also be illustrated in the form of

$$\tau_f = \{(a, \mathcal{A}(a)e^{i2\mathcal{P}(a)\pi}) : a \in \chi\}$$

Whereas mappings $\mathcal{A}: \chi \to [0,1]$ and $\mathcal{P}: \chi \to [0,1]$ refer to an amplitude term and phase term from which a membership grade is assigned to each element of set respectively.

Example 2. The set $\tau_f = \left\{ \begin{matrix} (a_\chi, 0.32e^{i2\pi(0.43)}), (b_\chi, 0.71e^{i2\pi(0.24)}), (c_\chi, 0.59e^{i2\pi(0.13)}), \\ (d_\chi, 0.29e^{i2\pi(0.41)}), (e_\chi, 0.97e^{i2\pi(0.33)}) \end{matrix} \right\}$ represents CFS.

Definition 3. [12] A set τ_f on a universal set χ, known as intuitionistic fuzzy set is of the following form

$$\tau_f\ = \{(a, \mathfrak{M}(a), \mathfrak{N}(a)) \colon a \in \chi\}$$

Whereas mappings $\mathfrak{M}\chi \to [0,1]$ and $\mathfrak{N} \colon \chi \to [0,1]$ assign membership and non-membership grades to each element of a set respectively. Moreover,

$$\mathfrak{M}(a) + \mathfrak{N}(a) \in [0,1].$$

Example 3. The set $\tau_f\ = \left\{\begin{matrix}(a_\chi, 0.42, 0.21), (b_\chi, 0.65, 0.33), (c_\chi, 0.72, 0.11), \\ (d_\chi, 0.18, 0.55), (e_\chi, 0.86, 0.023)\end{matrix}\right\}$ represents IFS.

Definition 4. [14] A set τ_f on a universal set χ, known as complex intuitionistic fuzzy set is of the following form

$$\tau_f\ = \left\{\left(a, \mathfrak{M}_{\mathbb{C}}(a), \mathfrak{N}_{\mathbb{C}}(a)\right) \colon a \in \chi\right\}$$

Whereas mappings $\mathfrak{M}_{\mathbb{C}} \colon \chi \to \mathbb{Z}$ and $\mathfrak{N}_{\mathbb{C}} \colon \chi \to \mathbb{Z} \ni 0 \le |\mathbb{Z}| \le 1$ assign membership and non-membership grades to each element of a set and $\mathbb{Z}$ is a complex number. Moreover,

$$|\mathfrak{M}_{\mathbb{C}}(a)| + |\mathfrak{N}_{\mathbb{C}}(a)| \in [0,1].$$

Furthermore, the complex intuitionistic fuzzy set can also be illustrated in the form of

$$\tau_f\ = \left\{\left(a, \mathcal{A}_{(\tau_f)\mathfrak{M}}(a) e^{i2\mathcal{P}_{\mathfrak{M}}(a)\pi}, \mathcal{A}_{(\tau_f)\mathfrak{N}}(a) e^{i2\mathcal{P}_{\mathfrak{N}}(a)\pi}\right) \colon a \in \chi\right\}$$

Whereas mappings $\mathcal{A}_{\mathfrak{M}} \colon \chi \to [0,1]$, $\mathcal{A}_{\mathfrak{N}} \colon \chi \to [0,1]$, $\mathcal{P}_{\mathfrak{M}} \colon \chi \to [0,1]$ and $\mathcal{P}_{\mathfrak{N}} \colon \chi \to [0,1]$ refer to amplitude terms of membership and non-membership grades and phase terms of membership and non-membership grades from which a membership and non-membership grades are assigned to each element of set respectively. Moreover,

$$(\mathcal{A}_{(\tau_f)\mathfrak{M}} + \mathcal{A}_{(\tau_f)\mathfrak{N}}) \in [0,1] \text{ and } (\mathcal{P}_{(\tau_f)\mathfrak{M}} + \mathcal{P}_{(\tau_f)\mathfrak{N}}) \in [0,1]$$

Example 4. The set

$$\tau_f\ = \left\{\begin{matrix}\left(a_\chi, 0.452e^{i2\pi(0.312)\pi}, 0.271e^{i2\pi(0.521)}\right), \left(b_\chi, 0.565e^{i2\pi(0.462)}, 0.323e^{i2\pi(0.113)}\right), \\ \left(c_\chi, 0.632e^{i2\pi(0.721)}, 0.311e^{i2\pi(0.213)}\right), \left(d_\chi, 0.128e^{i2\pi(0.299)}, 0.815e^{i2\pi(0.328)}\right), \\ \left(e_\chi, 0.486e^{i2\pi(0.235)}, 0.198e^{i2\pi(0.655)}\right)\end{matrix}\right\}$$

represents CIFS.

Definition 5. [26] A set τ_f on a universal set χ, known as complex Pythagorean fuzzy set is of the following form

$$\tau_f\ = \left\{\left(a, \mathfrak{M}_{\mathbb{C}}(a), \mathfrak{N}_{\mathbb{C}}(a)\right) \colon a \in \chi\right\}$$

Whereas mappings $\mathfrak{M}_{\mathbb{C}} \colon \chi \to \mathbb{Z}$ and $\mathfrak{N}_{\mathbb{C}} \colon \chi \to \mathbb{Z} \ni 0 \le |\mathbb{Z}| \le 1$ assigns a membership and non-membership to each element of set respectively and $\mathbb{Z}$ is a complex number.

Given that,

$$|\mathfrak{M}_{\mathbb{C}}(a)|^2 + |(\mathfrak{N}_{\mathbb{C}}(a)|^2 \in [0,1]$$

Furthermore, the complex Pythagorean fuzzy set can also be illustrated in the form of

$$\tau_f\ = \left\{\left(a, \mathcal{A}_{(\tau_f)\mathfrak{M}}(a) e^{i2\mathcal{P}_{(\tau_f)\mathfrak{M}}(a)\pi}, \mathcal{A}_{(\tau_f)\mathfrak{N}}(a) e^{i2\mathcal{P}_{(\tau_f)\mathfrak{N}}(a)\pi}\right) \colon a \in \chi\right\}$$

Whereas mappings $\mathcal{A}_{\mathfrak{M}} \colon \chi \to [0,1]$, $\mathcal{A}_{\mathfrak{N}} \colon \chi \to [0,1]$, $\mathcal{P}_{\mathfrak{M}} \colon \chi \to [0,1]$ and $\mathcal{P}_{\mathfrak{N}} \colon \chi \to [0,1]$ refer to amplitude terms of membership and non-membership and phase terms of membership and non-membership from which a membership and non-membership grades are assigned to each element of set respectively. Moreover,

$$\left(\mathcal{A}_{(\tau_f)\mathfrak{M}}\right)^2 + \left(\mathcal{A}_{(\tau_f)\mathfrak{N}}\right)^2 \in [0,1] \text{ an } \left(\mathcal{P}_{(\tau_f)\mathfrak{M}}\right)^2 + \left(\mathcal{P}_{(\tau_f)\mathfrak{N}}\right)^2 \in [0,1]$$

Example 5. The set

$$\tau_f = \left\{ \begin{matrix} \left(a_\chi, 0.752e^{i2\pi(0.512)\pi}, 0.471e^{i2\pi(0.671)}\right), \left(b_\chi, 0.575e^{i2\pi(0.462)}, 0.623e^{i2\pi(0.713)}\right), \\ \left(c_\chi, 0.652e^{i2\pi(0.771)}, 0.461e^{i2\pi(0.341)}\right), \\ \left(d_\chi, 0.128e^{i2\pi(0.799)}, 0.915e^{i2\pi(0.398)}\right), \left(e_\chi, 0.586e^{i2\pi(0.835)}, 0.498e^{i2\pi(0.255)}\right) \end{matrix} \right\}$$

represents CIFS.

Definition 6. [34] A set τ_f on a universal set χ, known as complex q-Rung orthopair fuzzy set is of the following form

$$\tau_f = \{(a, \mathfrak{M}_{\mathbb{C}}(a), \mathfrak{N}_{\mathbb{C}}(a)) : a \in \chi\}$$

Whereas mappings $\mathfrak{M}_{\mathbb{C}} : \chi \to \mathbb{Z}$ and $\mathfrak{N}_{\mathbb{C}} : \chi \to \mathbb{Z} \ni 0 \leq |\mathbb{Z}| \leq 1$ assign membership and non-membership to each element of set respectively and $\mathbb{Z}$ is a complex number.

Given that,

$$|\mathfrak{M}_{\mathbb{C}}(a)|^k + |(\mathfrak{N}_{\mathbb{C}}(a)|^k \in [0,1],\ where\ k = 3,4,5, \dots$$

Furthermore, the complex q-rung orthopair fuzzy set can also be illustrated in the form of

$$\tau_f = \left\{\left(a, \mathcal{A}_{(\tau_f)\mathfrak{M}}(a)e^{i2\mathcal{P}_{(\tau_f)\mathfrak{M}}(a)\pi}, \mathcal{A}_{(\tau_f)\mathfrak{N}}(a)e^{i2\mathcal{P}_{(\tau_f)\mathfrak{N}}(a)\pi}\right) : a \in \chi\right\}$$

Whereas mappings $\mathcal{A}_{\mathfrak{M}} : \chi \to [0,1]$, $\mathcal{A}_{\mathfrak{N}} : \chi \to [0,1]$, $\mathcal{P}_{\mathfrak{M}} : \chi \to [0,1]$ and $\mathcal{P}_{\mathfrak{N}} : \chi \to [0,1]$ refer to amplitude terms of membership and non-membership and phase terms of membership and non-membership from which a membership and non-membership grades are assigned to each element of set respectively. Moreover,

$$(\mathcal{A}_{(\tau_f)\mathfrak{M}})^k + (\mathcal{A}_{(\tau_f)\mathfrak{N}})^k \in [0,1] \text{ and } (\mathcal{P}_{(\tau_f)\mathfrak{M}})^k + (\mathcal{P}_{(\tau_f)\mathfrak{N}})^k \in [0,1], where\ k = 3,4,5, \dots$$

Example 6. The set

$$\tau_f = \left\{ \begin{matrix} \left(a_\chi, 0.852e^{i2\pi(0.812)\pi}, 0.571e^{i2\pi(0.679)}\right), \left(b_\chi, 0.775e^{i2\pi(0.862)}, 0.723e^{i2\pi(0.713)}\right), \\ \left(c_\chi, 0.792e^{i2\pi(0.871)}, 0.861e^{i2\pi(0.541)}\right), \left(d_\chi, 0.828e^{i2\pi(0.993)}, 0.965e^{i2\pi(0.498)}\right), \\ \left(e_\chi, 0.996e^{i2\pi(0.835)}, 0.678e^{i2\pi(0.555)}\right) \end{matrix} \right\}$$

represents CqROFS where $k = 9$.

Definition 7. [41] A set τ_f on a universal set χ, known as linear Diophantine fuzzy set is of the following form

$$\tau_f = \{(a, ((\mathfrak{M}_{(\tau_f)}(a), \mathfrak{N}_{(\tau_f)}(a)), (\mathfrak{A}, \mathfrak{B})) : a \in \chi\}$$

Whereas mappings $\mathfrak{M} : \chi \to [0,1]$ and $\mathfrak{N} : \chi \to [0,1]$ assign membership and non-membership grades and $\mathfrak{A}, \mathfrak{B} \in [0,1]$ assign reference parameters to each element of a set respectively. Given that

$$\left(\mathfrak{A}\mathfrak{M}(a) + \mathfrak{B}\mathfrak{N}(a)\right) \in [0,1] \text{ and } (\mathfrak{A} + \mathfrak{B}) \in [0,1]$$

Example 7. The set

$$\tau_f = \left\{ \begin{matrix} \left(a_\chi, (0.712, 0.421), (0.522, 0.281)\right), \left(b_\chi, (0.635, 0.373), (0.622, 0.261)\right), \left(c_\chi, (0.862, 0.311), (0.442, 0.521)\right), \\ \left(d_\chi, (0.928, 0.545), (0.452, 0.421)\right), \left(e_\chi, (0.286, 0.723), (0.482, 0.291)\right) \end{matrix} \right\}$$

represents LDFS.

3. Complex linear Diophantine fuzzy sets and relations

This section put forward definition of complex linear Diophantine fuzzy set and some new ideas related to the cartesian product of two CLDFSs, complex linear Diophantine fuzzy relation and its types. A suitable example is provided for every definition. Additionally, some intriguing outcomes for CLDFRs have also been established.

*Definition 8***.** A set τ_f on a universal set χ, known as complex linear Diophantine fuzzy set is of the following form

$$\tau_f = \left\{ \begin{matrix} (a, \left(\mathcal{A}_{(\tau_f)\mathfrak{M}}(a) e^{i2(\mathcal{P}_{(\tau_f)\mathfrak{M}}(a))\pi}, \mathcal{A}_{(\tau_f)\mathfrak{N}}(a) e^{i2\left(\mathcal{P}_{(\tau_f)\mathfrak{N}}(a)\right)\pi} \right), \\ \left((\mathcal{A}_{(\tau_f)\mathfrak{A}} e^{i2(\mathcal{P}_{(\tau_f)\mathfrak{A}})\pi}, \mathcal{A}_{(\tau_f)\mathfrak{B}} e^{i2\left(\mathcal{P}_{(\tau_f)\mathfrak{B}}\right)\pi} \right), : a \in \chi \end{matrix} \right\}$$

Whereas mapping $\mathcal{A}_{\mathfrak{M}}: \chi \to [0,1]$, $\mathcal{A}_{\mathfrak{N}}: \chi \to [0,1]$, $\mathcal{P}_{\mathfrak{M}}: \chi \to [0,1]$ and $\mathcal{P}_{\mathfrak{N}}: \chi \to [0,1]$ refer to amplitude terms of membership and non-membership and phase terms of membership and non-membership from which a membership and non-membership grades are assigned to each element of set respectively and $\mathcal{A}_{(\tau_f)\mathfrak{A}}, \mathcal{A}_{(\tau_f)\mathfrak{B}}, \mathcal{P}_{(\tau_f)\mathfrak{A}}, \mathcal{P}_{(\tau_f)\mathfrak{B}} \in [0,1]$ refer to amplitude and phase terms of reference parameters such that

$$\left(\mathcal{A}_{(\tau_f)\mathfrak{A}} + \mathcal{A}_{(\tau_f)\mathfrak{B}}\right) \in [0,1], \left(\mathcal{A}_{(\tau_f)\mathfrak{A}}\mathcal{A}_{(\tau_f)\mathfrak{M}}(a) + \mathcal{A}_{(\tau_f)\mathfrak{B}}\mathcal{A}_{(\tau_f)\mathfrak{N}}(a)\right) \in [0,1] \text{ and}$$

$$\left(\mathcal{P}_{(\tau_f)\mathfrak{A}} + \mathcal{P}_{(\tau_f)\mathfrak{B}}\right) \in [0,1], \left(\mathcal{P}_{(\tau_f)\mathfrak{A}}\mathcal{P}_{(\tau_f)\mathfrak{M}}(a) + \mathcal{P}_{(\tau_f)\mathfrak{B}}\mathcal{P}_{(\tau_f)\mathfrak{N}}(a)\right) \in [0,1]$$

Example 8. The set

$$\tau_f = \left\{ \begin{matrix} \begin{pmatrix} a_\chi, (0.521e^{i2\pi(0.692)}, 0.472e^{i2\pi(0.317)}), \\ (0.523e^{i2\pi(0.433)}, 0.347e^{i2\pi(0.329)}) \end{pmatrix}, \begin{pmatrix} b_\chi, (0.717e^{i2\pi(0.511)}, 0.521e^{i2\pi(0.612)}), \\ (0.646e^{i2\pi(0.363)}, 0.325e^{i2\pi(0.413)}) \end{pmatrix}, \\ \begin{pmatrix} c_\chi, (0.648e^{i2\pi(0.709)}, 0.331e^{i2\pi(0.471)}), \\ (0.369e^{i2\pi(0.401)}, 0.573e^{i2\pi(0.395)}) \end{pmatrix} \end{matrix} \right\}$$

represents CLDFS.

Definition 9. Let

$$\tau_f = \left\{ \begin{matrix} (a, \left(\mathcal{A}_{(\tau_f)\mathfrak{M}}(a) e^{i2(\mathcal{P}_{(\tau_f)\mathfrak{M}}(a))\pi}, \mathcal{A}_{(\tau_f)\mathfrak{N}}(a) e^{i2\left(\mathcal{P}_{(\tau_f)\mathfrak{N}}(a)\right)\pi} \right), \\ \left((\mathcal{A}_{(\tau_f)\mathfrak{A}} e^{i2(\mathcal{P}_{(\tau_f)\mathfrak{A}})\pi}, \mathcal{A}_{(\tau_f)\mathfrak{B}} e^{i2\left(\mathcal{P}_{(\tau_f)\mathfrak{B}}\right)\pi} \right), : a \in \chi) \end{matrix} \right\} \text{ and}$$

$$\nu_f = \left\{ \left(\begin{matrix} b, \left(\mathcal{A}_{(\nu_f)\mathfrak{M}}(b) e^{i2(\mathcal{P}_{(\nu_f)\mathfrak{M}}(b))\pi}, \mathcal{A}_{(\nu_f)\mathfrak{N}}(b) e^{i2\left(\mathcal{P}_{(\nu_f)\mathfrak{N}}(b)\right)\pi} \right), \\ \left((\mathcal{A}_{(\nu_f)\mathfrak{A}} e^{i2(\mathcal{P}_{(\nu_f)\mathfrak{A}})\pi}, \mathcal{A}_{(\nu_f)\mathfrak{B}} e^{i2\left(\mathcal{P}_{(\nu_f)\mathfrak{B}}\right)\pi} \right), : b \in \chi \end{matrix} \right) \right\}$$

be two CLDFSs in a universal set χ, then their cartesian product is given as $\tau_f \times \nu_f$

$$= \left\{ \left(\begin{matrix} (a,b), \left(\mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{M}}(a,b) e^{i2\left(\mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{M}}(a,b)\right)\pi}, \mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{N}}(a,b) e^{i2\left(\mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{N}}(a,b)\right)\pi} \right), \\ \left(\mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{A}}, e^{i2\left(\mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{A}}\right)\pi}, \mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{B}} e^{i2\left(\mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{B}}\right)\pi} \right) \end{matrix} \right) \\ : a \in \tau_f, b \in \nu_f \right\}$$

Whereas

$\mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{M}}(a,b) = \min\{\mathcal{A}_{(\tau_f)\mathfrak{M}}(a), \mathcal{A}_{(\nu_f)\mathfrak{M}}(b)\}, \mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{M}}(a,b) = \min\left\{\mathcal{P}_{(\tau_f)\mathfrak{M}}(a), \mathcal{P}_{(\nu_f)\mathfrak{M}}(b)\right\},$

$\mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{N}}(a,b) = \max\left\{\mathcal{A}_{(\tau_f)\mathfrak{N}}(a), \mathcal{A}_{(\nu_f)\mathfrak{N}}(b)\right\}, \mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{N}}(a,b) = \max\left\{\mathcal{P}_{(\tau_f)\mathfrak{N}}(a), \mathcal{P}_{(\nu_f)\mathfrak{N}}(b)\right\}$ and

$\mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{A}} = min\left\{\mathcal{A}_{(\tau_f)\mathfrak{A}}, \mathcal{A}_{(\nu_f)\mathfrak{A}}\right\}, \mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{A}} = min\left\{\mathcal{P}_{(\tau_f)\mathfrak{A}}, \mathcal{A}_{(\nu_f)\mathfrak{A}}\right\},$

$\mathcal{A}_{(\tau_f \times \nu_f)\mathfrak{B}} = max\left\{\mathcal{A}_{(\tau_f)\mathfrak{B}}, \mathcal{A}_{(\nu_f)\mathfrak{B}}\right\}, \mathcal{P}_{(\tau_f \times \nu_f)\mathfrak{B}} = max\left\{\mathcal{P}_{(\tau_f)\mathfrak{B}}, \mathcal{P}_{(\nu_f)\mathfrak{B}}\right\}.$

Definition 10. The complex linear Diophantine fuzzy relation is a subset of the cartesian product of any two CLDFSs, $\bar{\mathcal{R}} \subseteq \tau_f \times \nu_f$, where τ_f and ν_f are CLDFSs and $\bar{\mathcal{R}}$ denotes the CLDFR.

Example 10. The cartesian product of two CLDFSs

$$\tau_f = \left\{\begin{matrix} \begin{pmatrix} a, (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.217)}), \\ (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)}) \end{pmatrix}, \\ \begin{pmatrix} b, (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.212)}), \\ (0.346e^{i2\pi(0.333)}, 0.315e^{i2\pi(0.213)}) \end{pmatrix}, \\ \begin{pmatrix} c, (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), \\ (0.369e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)}) \end{pmatrix} \end{matrix}\right\} \text{ and}$$

$$\nu_f = \left\{\begin{matrix} \begin{pmatrix} l, (0.313e^{i2\pi(0.323)}, 0.435e^{i2\pi(0.381)}), \\ (0.296e^{i2\pi(0.357)}, 0.374e^{i2\pi(0.329)}) \end{pmatrix}, \\ \begin{pmatrix} m, (0.404e^{i2\pi(0.332)}, 0.297e^{i2\pi(0.340)}), \\ (0.244e^{i2\pi(0.353)}, 0.607e^{i2\pi(0.359)}) \end{pmatrix}, \\ \begin{pmatrix} n, (0.242e^{i2\pi(0.349)}, 0.367e^{i2\pi(0.492)}), \\ (0.337e^{i2\pi(0.287)}, 0.399e^{i2\pi(0.395)}) \end{pmatrix} \end{matrix}\right\}$$

is

$$\tau_f \times \nu_f = \left\{\begin{matrix} \left((a,l), (0.313e^{i2\pi(0.292)}, 0.435e^{i2\pi(0.381)}), (0.123e^{i2\pi(0.233)}, 0.374e^{i2\pi(0.329)})\right), \\ \left((a,m), (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.3340)}), (0.123e^{i2\pi(0.233)}, 0.607e^{i2\pi(0.359)})\right), \\ \left((a,n), (0.242e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.492)}), (0.123e^{i2\pi(0.233)}, 0.399e^{i2\pi(0.395)})\right), \\ \left((b,l), (0.313e^{i2\pi(0.211)}, 0.435e^{i2\pi(0.381)}), (0.296e^{i2\pi(0.333)}, 0.374e^{i2\pi(0.329)})\right), \\ \left((b,m), (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.340)}), (0.244e^{i2\pi(0.333)}, 0.607e^{i2\pi(0.359)})\right), \\ \left((b,n), (0.242e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.492)}), (0.337e^{i2\pi(0.287)}, 0.399e^{i2\pi(0.395)})\right), \\ \left((c,l), (0.313e^{i2\pi(0.209)}, 0.435e^{i2\pi(0.381)}), (0.296e^{i2\pi(0.301)}, 0.374e^{i2\pi(0.345)})\right), \\ \left((c,m), (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), (0.244e^{i2\pi(0.301)}, 0.607e^{i2\pi(0.359)})\right), \\ \left((c,n), (0.242e^{i2\pi(0.209)}, 0.367e^{i2\pi(0.492)}), (0.337e^{i2\pi(0.287)}, 0.399e^{i2\pi(0.395)})\right) \end{matrix}\right\}$$

The CLDFR $\bar{\mathcal{R}}$ among the CLDFSs τ_f and ν_f is

$$\bar{\mathcal{R}} = \left\{ \begin{array}{c} \big((a,m),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.3340)}),(0.123e^{i2\pi(0.233)},0.607e^{i2\pi(0.359)})\big), \\ \big((b,l),(0.313e^{i2\pi(0.211)},0.435e^{i2\pi(0.381)}),(0.296e^{i2\pi(0.333)},0.374e^{i2\pi(0.329)})\big), \\ \big((b,n),(0.242e^{i2\pi(0.211)},0.421e^{i2\pi(0.492)}),(0.337e^{i2\pi(0.287)},0.399e^{i2\pi(0.395)})\big), \\ \big((c,m),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.244e^{i2\pi(0.301)},0.607e^{i2\pi(0.359)})\big), \\ \big((c,n),(0.242e^{i2\pi(0.209)},0.367e^{i2\pi(0.492)}),(0.337e^{i2\pi(0.287)},0.399e^{i2\pi(0.395)})\big) \end{array} \right\}$$

Definition 11. Let τ_f be an CLDFS in a universal set χ and $\bar{\mathcal{R}}$ be an CLDFR on τ_f. Then

i. If $(a,a) \in \bar{\mathcal{R}}, \forall a \in \tau_f$, then $\bar{\mathcal{R}}$ is referred to as a CLD reflexive fuzzy relation (CLD-reflexive-FR) on τ_f .
ii. If $(a,a) \notin \bar{\mathcal{R}}$, $\forall a \in \tau_f$, then $\bar{\mathcal{R}}$ is referred to as a CLD irreflexive fuzzy relation (CLD-irreflexive-FR) on τ_f .
iii. If $(a,b) \in \bar{\mathcal{R}} \Rightarrow (b,a) \in \bar{\mathcal{R}}$, $\forall a,b \in \tau_f$, then $\bar{\mathcal{R}}$ is referred to as a CLD symmetric fuzzy relation (CLD-symmetric-FR) on τ_f .
iv. If $(a,b) \in \bar{\mathcal{R}}$ and $(b,a) \in \bar{\mathcal{R}} \Rightarrow a = b$, $\forall a,b \in \tau_f$, then $\bar{\mathcal{R}}$ is referred to as a CLD antisymmetric fuzzy relation (CLD-antisymmetric-FR) on τ_f .
v. If $(b,a) \in \bar{\mathcal{R}} \Rightarrow (a = b \notin)\bar{\mathcal{R}}$, $\forall a,b \in \tau_f$, then $\bar{\mathcal{R}}$ is referred to as a CLD asymmetric fuzzy relation (CLD-asymmetric-FR) on τ_f .
vi. If $(a,b) \in \bar{\mathcal{R}}$ or $(b,a) \in \bar{\mathcal{R}}$, $\forall a,b \in \tau_f$, then $\bar{\mathcal{R}}$ is referred to as a CLD complete fuzzy relation (CLD-complete-FR) on τ_f .
vii. If $(a,b) \in \bar{\mathcal{R}}$ and $(b,c) \in \bar{\mathcal{R}} \Rightarrow (a,c) \in \bar{\mathcal{R}}$, $\forall a,b,c \in \tau_f$, then $\bar{\mathcal{R}}$ is referred to as a CLD transitive fuzzy relation (CLD-transitive-FR) on τ_f .
viii. If $\bar{\mathcal{R}}$ is CLD-reflexive-FR, CLD-symmetric-FR and CLD-transitive-FR on τ_f , then $\bar{\mathcal{R}}$ is referred to as a CLD equivalence fuzzy relation (CLD-equivalence-FR) on τ_f .
ix. If $\bar{\mathcal{R}}$ is CLD-reflexive-FR and CLD-transitive-FR on τ_f, then $\bar{\mathcal{R}}$ is referred to as a CLD preorder fuzzy relation (CLD- preorder-FR) on τ_f .
x. If $\bar{\mathcal{R}}$ is CLD-irreflexive-FR and CLD-transitive-FR on τ_f, then $\bar{\mathcal{R}}$ is referred to as a CLD strict order fuzzy relation (CLD- strict order-FR) on τ_f.
xi. If $\bar{\mathcal{R}}$ is CLD-preorder-FR and CLD-antisymmetric-FR on τ_f, then $\bar{\mathcal{R}}$ is referred to as a CLD partial order fuzzy relation (CLD- partial order-FR) on τ_f.
xii. If $\bar{\mathcal{R}}$ is CLD-partial order-FR and CLD-complete-FR on τ_f, then $\bar{\mathcal{R}}$ is referred to as a CLD linear order fuzzy relation (CLD- linear order-FR) on τ_f.

Example 11. For an CLDFS

$$\tau_f = \left\{ \begin{array}{c} \begin{pmatrix} a,(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}), \\ (0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)}) \end{pmatrix}, \\ \begin{pmatrix} b,(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}), \\ (0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)}) \end{pmatrix}, \\ \begin{pmatrix} c,(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}), \\ (0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)}) \end{pmatrix} \end{array} \right\}$$

the cartesian product of $\tau_f \times \tau_f$ is

$$\tau_f \times \tau_f = \left\{ \begin{array}{l} \Big((a,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((a,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((a,c),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((b,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}),(0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)})\Big), \\ \Big((b,a),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((b,c),(0.317e^{i2\pi(0.209)},0.421e^{i2\pi(0.371)}),(0.346e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((c,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((c,a),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((c,b),(0.317e^{i2\pi(0.209)},0.421e^{i2\pi(0.371)}),(0.346e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\Big) \end{array} \right\}$$

Then,

i. The CLD-equivalence-FR $\bar{\mathcal{R}}_1$on τ_f is as follow

$$\bar{\mathcal{R}}_1 = \left\{ \begin{array}{l} \Big((a,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((a,c),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((b,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}),(0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)})\Big), \\ \Big((c,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((c,a),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\Big) \end{array} \right\}$$

ii. The CLD-preorder-FR $\bar{\mathcal{R}}_2$ on τ_f is as follow

$$\bar{\mathcal{R}}_2 = \left\{ \begin{array}{l} \Big((a,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((b,a),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((b,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}),(0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)})\Big), \\ \Big((c,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\Big) \end{array} \right\}$$

iii. The CLD-strict order-FR $\bar{\mathcal{R}}_3$ on τ_f is as follow

$$\bar{\mathcal{R}}_3 = \left\{ \begin{array}{l} \Big((b,a),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((c,a),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((c,b),(0.317e^{i2\pi(0.209)},0.421e^{i2\pi(0.371)}),(0.346e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\Big) \end{array} \right\}$$

iv. The CLD-partial order-FR $\bar{\mathcal{R}}_4$ on τ_f is as follow

$$\bar{\mathcal{R}}_4 = \left\{ \begin{array}{l} \Big((a,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((b,a),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\Big), \\ \Big((b,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}),(0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)})\Big), \\ \Big((c,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\Big), \\ \Big((c,a),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\Big) \end{array} \right\}$$

v. The CLD-linear order-FR $\bar{\mathcal{R}}_5$ on τ_f is as follow

$$\bar{\mathcal{R}}_5=\left\{\begin{array}{c}\left((a,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\right),\\ \left((b,a),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\right),\\ \left((b,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}),(0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)})\right),\\ \left((c,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\right),\\ \left((c,a),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\right),\\ \left((c,b),(0.317e^{i2\pi(0.209)},0.421e^{i2\pi(0.371)}),(0.346e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\right)\end{array}\right\}$$

Definition 12. The converse relation $\bar{\mathcal{R}}^c$ for CLDFR $\bar{\mathcal{R}}$ is defined as,

$$\bar{\mathcal{R}}^c=\{(b,a):(a,b)\in\bar{\mathcal{R}}\}$$

Example 12. The converse relation $\bar{\mathcal{R}}^c$for CLDFR

$$\bar{\mathcal{R}}=\left\{\begin{array}{c}\left((a,m),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.3340)}),(0.123e^{i2\pi(0.233)},0.607e^{i2\pi(0.359)})\right),\\ \left((b,l),(0.313e^{i2\pi(0.211)},0.435e^{i2\pi(0.381)}),(0.296e^{i2\pi(0.333)},0.374e^{i2\pi(0.329)})\right),\\ \left((b,n),(0.242e^{i2\pi(0.211)},0.421e^{i2\pi(0.492)}),(0.337e^{i2\pi(0.287)},0.399e^{i2\pi(0.395)})\right),\\ \left((c,m),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.244e^{i2\pi(0.301)},0.607e^{i2\pi(0.359)})\right),\\ \left((c,n),(0.242e^{i2\pi(0.209)},0.367e^{i2\pi(0.492)}),(0.337e^{i2\pi(0.287)},0.399e^{i2\pi(0.395)})\right)\end{array}\right\}$$

Such that $\bar{\mathcal{R}}$ is a CLDFR between CLDFSs

$$\tau_f=\left\{\begin{array}{c}\begin{pmatrix}a,(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),\\ (0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\end{pmatrix},\\ \begin{pmatrix}b,(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}),\\ (0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)})\end{pmatrix},\\ \begin{pmatrix}c,(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),\\ (0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\end{pmatrix}\end{array}\right\}\text{ and}$$

$$\nu_f=\left\{\begin{array}{c}\begin{pmatrix}l,(0.313e^{i2\pi(0.323)},0.435e^{i2\pi(0.381)}),\\ (0.296e^{i2\pi(0.357)},0.374e^{i2\pi(0.329)})\end{pmatrix},\\ \begin{pmatrix}m,(0.404e^{i2\pi(0.332)},0.297e^{i2\pi(0.340)}),\\ (0.244e^{i2\pi(0.353)},0.607e^{i2\pi(0.359)})\end{pmatrix},\\ \begin{pmatrix}n,(0.242e^{i2\pi(0.349)},0.367e^{i2\pi(0.492)}),\\ (0.337e^{i2\pi(0.287)},0.399e^{i2\pi(0.395)})\end{pmatrix}\end{array}\right\}\text{ is given as,}$$

$$\bar{\mathcal{R}}^c=\left\{\begin{array}{c}\left((m,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.3340)}),(0.123e^{i2\pi(0.233)},0.607e^{i2\pi(0.359)})\right),\\ \left((l,b),(0.313e^{i2\pi(0.211)},0.435e^{i2\pi(0.381)}),(0.296e^{i2\pi(0.333)},0.374e^{i2\pi(0.329)})\right),\\ \left((n,b),(0.242e^{i2\pi(0.211)},0.421e^{i2\pi(0.492)}),(0.337e^{i2\pi(0.287)},0.399e^{i2\pi(0.395)})\right),\\ \left((m,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.244e^{i2\pi(0.301)},0.607e^{i2\pi(0.359)})\right),\\ \left((n,c),(0.242e^{i2\pi(0.209)},0.367e^{i2\pi(0.492)}),(0.337e^{i2\pi(0.287)},0.399e^{i2\pi(0.395)})\right)\end{array}\right\}$$

The CLD-equivalence-FRs generate the concept of CLD-equivalence classes, which are described as follows.

Definition 13. Let $\bar{\mathcal{R}}$ is an CLD-equivalence-FR, then CLDF-equivalence class of $a\ modulo\ \bar{\mathcal{R}}$ is defined as, $\bar{\mathcal{R}}[a] = \{b \mid (b, a) \in \bar{\mathcal{R}}\}$.

Example 13. Let

$$\bar{\mathcal{R}} = \left\{ \begin{array}{l} \Big((a,a), (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.217)}), (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)})\Big), \\ \Big((a,c), (0.321e^{i2\pi(0.209)}, 0.372e^{i2\pi(0.371)}), (0.123e^{i2\pi(0.233)}, 0.373e^{i2\pi(0.345)})\Big), \\ \Big((b,b), (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.212)}), (0.346e^{i2\pi(0.333)}, 0.315e^{i2\pi(0.213)})\Big), \\ \Big((c,c), (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), (0.369e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)})\Big), \\ \Big((c,a), (0.321e^{i2\pi(0.209)}, 0.372e^{i2\pi(0.371)}), (0.123e^{i2\pi(0.233)}, 0.373e^{i2\pi(0.345)})\Big) \end{array} \right\}$$

is an CLD-equivalence-FR on an CLDFS

$$\tau_f = \left\{ \begin{array}{l} \begin{pmatrix} a, (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.217)}), \\ (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)}) \end{pmatrix}, \\ \begin{pmatrix} b, (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.212)}), \\ (0.346e^{i2\pi(0.333)}, 0.315e^{i2\pi(0.213)}) \end{pmatrix}, \\ \begin{pmatrix} c, (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), \\ (0.369e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)}) \end{pmatrix} \end{array} \right\}$$

Then CLDF-equivalence class of

i. $a\ modulo\ \bar{\mathcal{R}}$ is given as

$$\bar{\mathcal{R}}[a] = \left\{ \begin{array}{l} \begin{pmatrix} a, (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.217)}), \\ (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)}) \end{pmatrix}, \\ \begin{pmatrix} c, (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), \\ (0.369e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)}) \end{pmatrix} \end{array} \right\}$$

ii. $b\ modulo\ \bar{\mathcal{R}}$ is given as

$$\bar{\mathcal{R}}[b] = \left\{ \begin{pmatrix} b, (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.212)}), \\ (0.346e^{i2\pi(0.333)}, 0.315e^{i2\pi(0.213)}) \end{pmatrix} \right\}$$

iii. $c\ modulo\ \bar{\mathcal{R}}$ is given as

$$\bar{\mathcal{R}}[c] = \left\{ \begin{array}{l} \begin{pmatrix} a, (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.217)}), \\ (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)}) \end{pmatrix}, \\ \begin{pmatrix} c, (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), \\ (0.369e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)}) \end{pmatrix} \end{array} \right\}$$

Definition 14. Let $\bar{\mathcal{R}}$ be an CLDFR on an CLDFS τ_f, then CLD-composite-FR $\bar{\mathcal{R}} \circ \bar{\mathcal{R}}$ is defined as, $\forall\ (a, b) \in \bar{\mathcal{R}}$ and $(b, c) \in \bar{\mathcal{R}} \Rightarrow (a, c) \in \bar{\mathcal{R}} \circ \bar{\mathcal{R}}, \forall a, b, c \in \chi$.

Example 14. Let $\bar{\mathcal{R}}_1$ and $\bar{\mathcal{R}}_2$ be two CLDFR's such that

$$\bar{\mathcal{R}}_1 = \left\{ \begin{array}{l} \Big((a,b), (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.217)}), (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)})\Big), \\ \Big((b,a), (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.217)}), (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)})\Big), \\ \Big((c,b), (0.317e^{i2\pi(0.209)}, 0.421e^{i2\pi(0.371)}), (0.346e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)})\Big) \end{array} \right\}$$

and

$$\bar{\mathcal{R}}_2 = \left\{\begin{array}{l} \left((a,c),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\right), \\ \left((c,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\right), \\ \left((b,a),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\right), \\ \left((c,b),(0.317e^{i2\pi(0.209)},0.421e^{i2\pi(0.371)}),(0.346e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\right) \end{array}\right\}$$

on CLDFS $\tau_f = \left\{\begin{array}{l} \begin{pmatrix} a,(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}), \\ (0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)}) \end{pmatrix}, \\ \begin{pmatrix} b,(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}), \\ (0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)}) \end{pmatrix}, \\ \begin{pmatrix} c,(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}), \\ (0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)}) \end{pmatrix} \end{array}\right\},$

Then the CLD-composite-FR $\bar{\mathcal{R}}_1 \circ \bar{\mathcal{R}}_2$ is given as,

$$\bar{\mathcal{R}}_1 \circ \bar{\mathcal{R}}_2 = \left\{\begin{array}{l} \left((a,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\right), \\ \left((b,c),(0.317e^{i2\pi(0.209)},0.421e^{i2\pi(0.371)}),(0.346e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\right), \\ \left((c,a),(0.321e^{i2\pi(0.209)},0.372e^{i2\pi(0.371)}),(0.123e^{i2\pi(0.233)},0.373e^{i2\pi(0.345)})\right) \end{array}\right\}$$

Theorem 1. An CLDFR $\bar{\mathcal{R}}$ is an CLD-symmetric-FR on an CLDFS τ_f iff $\bar{\mathcal{R}} = \bar{\mathcal{R}}^c$.

Proof. Assume that $\bar{\mathcal{R}} = \bar{\mathcal{R}}^c$, then

$(a,b) \in \bar{\mathcal{R}} \Rightarrow (b,a) \in \bar{\mathcal{R}}^c \Rightarrow (b,a) \in \bar{\mathcal{R}}$.

Thus, $\bar{\mathcal{R}}$ is an CLD-symmetric-FR on an CLDFS τ_f .

Conversely, suppose that $\bar{\mathcal{R}}$ is an CLD-symmetric-FR on an CLDFS τ_f , then

$(a,b) \in \bar{\mathcal{R}} \Rightarrow (b,a) \in \bar{\mathcal{R}}$.

However, $(b,a) \in \bar{\mathcal{R}}^c \Rightarrow \bar{\mathcal{R}} = \bar{\mathcal{R}}^c$.

Theorem 2. An CLDFR $\bar{\mathcal{R}}$ is an CLD-transitive-FR on an CLDFS τ_f iff $\bar{\mathcal{R}} \circ \bar{\mathcal{R}} \subseteq \bar{\mathcal{R}}$.

Proof. Assume that $\bar{\mathcal{R}}$ is an CLD-transitive-FR on an CLDFS τ_f .

Let $(a,c) \in \bar{\mathcal{R}} \circ \bar{\mathcal{R}}$,

Then, by definition of CLD-transitive-FR,

$(a,b) \in \bar{\mathcal{R}}$ and $(b,c) \in \bar{\mathcal{R}} \Rightarrow (a,c) \in \bar{\mathcal{R}} \Rightarrow \bar{\mathcal{R}} \circ \bar{\mathcal{R}} \subseteq \bar{\mathcal{R}}$.

Conversely assume that $\bar{\mathcal{R}} \circ \bar{\mathcal{R}} \subseteq \bar{\mathcal{R}}$, then

For $(a,b) \in \bar{\mathcal{R}}$ and $(b,c) \in \bar{\mathcal{R}} \Rightarrow (a,c) \in \bar{\mathcal{R}} \circ \bar{\mathcal{R}} \subseteq \bar{\mathcal{R}} \Rightarrow (a,c) \in \bar{\mathcal{R}}$.

Thus, $\bar{\mathcal{R}}$ is an CLD-transitive-FR on an CLDFS τ_f.

Theorem 3. Suppose $\bar{\mathcal{R}}$ is an CLD-equivalence-FR on an CLDFS τ_f, then $\bar{\mathcal{R}} \circ \bar{\mathcal{R}} = \bar{\mathcal{R}}$.

Proof. Assume that $(a,b) \in \bar{\mathcal{R}}$,

Then by definition of CLD-symmetric-FR,

$(b,a) \in \bar{\mathcal{R}}$.

Now, by using the definition of CLD-transitive-FR,

$(a,a) \in \bar{\mathcal{R}}$.

However, by the definition of CLD-composite-FR,

$(a,a) \in \bar{\mathcal{R}} \circ \bar{\mathcal{R}}$.

Thus, $\bar{\mathcal{R}} \subseteq \bar{\mathcal{R}} \circ \bar{\mathcal{R}}$ (1)

Conversely, assume that $(a,b) \in \bar{\mathcal{R}} \circ \bar{\mathcal{R}}$, then $\exists\ c \in U \ni (a,c) \in \bar{\mathcal{R}}$ and $(c,b) \in \bar{\mathcal{R}}$.

However, it is given that $\bar{\mathcal{R}}$ is an CLD-equivalence-FR on CLDFS τ_f , so $\bar{\mathcal{R}}$ is also an CLD-transitive-FR. Therefore, $(a,b) \in \bar{\mathcal{R}} \Rightarrow \bar{\mathcal{R}} \circ \bar{\mathcal{R}} \subseteq \bar{\mathcal{R}}$ (2)

Thus, by (1) and (2),

$$\bar{\mathcal{R}} \circ \bar{\mathcal{R}} = \bar{\mathcal{R}}$$

Theorem 4. Suppose $\bar{\mathcal{R}}$ is an CLD-partial order-FR on an CLDFS τ_f, then the converse relation $\bar{\mathcal{R}}^c$ of $\bar{\mathcal{R}}$ is also an CLD-partial order-FR on an CLDFS τ_f.

Proof. In order to prove the assertion, it is sufficient to show that the converse of a complex linear Diophantine partial order fuzzy relation $\bar{\mathcal{R}}^c$ satisfies the three properties of complex linear Diophantine partial order fuzzy relation.

By using the properties of CLD-partial order-FR $\bar{\mathcal{R}}$, we prove the statement.

i. It is given that $\bar{\mathcal{R}}$ is an CLD-reflexive-FR. Therefore, for any $a \in U, (a,a) \in \bar{\mathcal{R}} \Rightarrow (a,a) \in \bar{\mathcal{R}}^c$. Thus, $\bar{\mathcal{R}}^c$ is an CLD-reflexive-FR.
ii. Assume that $(a,a) \in \bar{\mathcal{R}}^c$ and $(b,a) \in \bar{\mathcal{R}}^c$, then, $(a,b) \in \bar{\mathcal{R}}$ and $(b,a) \in \bar{\mathcal{R}}$. However, $\bar{\mathcal{R}}$ is an CLD-antisymmetric-FR. Therefore, $(a,b) = (b,a)$. Thus, $\bar{\mathcal{R}}^c$ is an CLD-antisymmetric-FR.
iii. Suppose that $(a,b) \in \bar{\mathcal{R}}^c$ and $(b,c) \in \bar{\mathcal{R}}^c$, then, $(c,b) \in \bar{\mathcal{R}}$ and $(b,a) \in \bar{\mathcal{R}}$. However, it is given that $\bar{\mathcal{R}}$ is an CLD-transitive-FR. Therefore, $(c,a) \in \bar{\mathcal{R}} \Rightarrow (a,c) \in \bar{\mathcal{R}}^c$. Thus, $\bar{\mathcal{R}}^c$ is an CLD-transitive-FR.

From i, ii and iii, the converse relation $\bar{\mathcal{R}}^c$ of an CLD-partial order-FR $\bar{\mathcal{R}}$ is proved to be an CLD-partial order-FR too.

Theorem 5. Suppose $\bar{\mathcal{R}}$ is an CLD-equivalence-FR on an CLDFS τ_f, then $(a,b) \in \bar{\mathcal{R}},\ iff\ \bar{\mathcal{R}}[a] = \bar{\mathcal{R}}[b]$.

Proof. Assume that $(a,b) \in \bar{\mathcal{R}}$ and $c \in \bar{\mathcal{R}}[a] \Rightarrow (c,a) \in \bar{\mathcal{R}}$.

Now, by using the fact that an CLD-equivalence-FR is also an CLD-transitive-FR, so $(c,b) \in \bar{\mathcal{R}} \Rightarrow c \in \bar{\mathcal{R}}[b]$.

Thus, $\bar{\mathcal{R}}[a] \subseteq \bar{\mathcal{R}}[b]$ (3)

As $(a,b) \in \bar{\mathcal{R}}$, by using the fact that an CLD-equivalence-FR is also an CLD-symmetric-FR, so $(b,a) \in \bar{\mathcal{R}}$.

Additionally, assume that $c \in \bar{\mathcal{R}}[b] \Rightarrow (c,b) \in \bar{\mathcal{R}}$.

Now, again by using the fact that an CLD-equivalence-FR is also an CLD-transitive-FR, so

$$(c,a) \in \bar{\mathcal{R}} \Rightarrow c \in \bar{\mathcal{R}}[a]$$

Thus, $\bar{\mathcal{R}}[a] \supseteq \bar{\mathcal{R}}[b]$ (4)

Therefore, from (3) and (4)

$$\bar{\mathcal{R}}[a] = \bar{\mathcal{R}}[b]$$

Conversely, assume that $\bar{\mathcal{R}}[a] = \bar{\mathcal{R}}[b]$, $c \in \bar{\mathcal{R}}[a]$ and $c \in \bar{\mathcal{R}}[b] \Rightarrow (c,b) \in \bar{\mathcal{R}}$ and $(c,a) \in \bar{\mathcal{R}}$.

Again, by using the fact that an CLD-equivalence-FR is also an CLD-symmetric-FR, so $(c,a) \in \bar{\mathcal{R}} \Rightarrow (a,c) \in \bar{\mathcal{R}}$.

Now, by definition of CLD-transitive-FR,

$(a,c) \in \bar{\mathcal{R}}$ and $(c,b) \in \bar{\mathcal{R}} \Rightarrow (a,b) \in \bar{\mathcal{R}}$.

Hence proved.

4. Hasse diagram for CLD-partial order-FRs

In this segment, we define the Hasse diagram for CLD-partial order-FR. In order to illustrate the CLD-partial order-FR framework, the Hasse diagram is essential. This diagram is a graphical tool that simplifies the understanding of complex relationships in such frameworks. In a Hasse diagram, elements of partially ordered set are represented by points (vertices), and the ordering relation between the element is represented by connecting line segment (edges). In constructing a Hasse diagram, certain rules are discussed below:

i. The elements are organized from lower to higher ranks. In any pair, the first element is smaller than the second. For instance, in $(a, b), a$ is smaller and is placed lower than b in the diagram.
ii. A self-relation is not represented by a line. Instead, its existence is understood without explicit representation.
iii. Unlike other diagrams, the Hasse diagram doesn't use arrows. The arrangement of element in relation to each other indicates their order, removing the need for directional indicators.
iv. Unnecessary lines are omitted in the diagram. For instance, when considering the element (a, b) and their transitive relationship with (a, c) only two lines are drawn: one from a to b and another from a to c. This approach makes indirect relationship easier to understand and simplifies the diagram.

Example 15. Consider CLDFS

$$\tau_f = \left\{ \begin{array}{l} \begin{pmatrix} a, (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.217)}), \\ (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)}) \end{pmatrix}, \\ \begin{pmatrix} b, (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.212)}), \\ (0.346e^{i2\pi(0.333)}, 0.315e^{i2\pi(0.213)}) \end{pmatrix}, \\ \begin{pmatrix} c, (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), \\ (0.369e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)}) \end{pmatrix}, \\ \begin{pmatrix} l, (0.313e^{i2\pi(0.323)}, 0.435e^{i2\pi(0.381)}), \\ (0.296e^{i2\pi(0.357)}, 0.374e^{i2\pi(0.329)}) \end{pmatrix} \end{array} \right\} \text{in a universal set } \chi.$$

Then the cartesian product $\tau_f \times \tau_f$ is given as

$$\tau_f \times \tau_f = \left\{ \begin{array}{l} \left((a, a), (0.321e^{i2\pi(0.292)}, 0.372e^{i2\pi(0.217)}), (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)})\right), \\ \left((a, b), (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.217)}), (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)})\right), \\ \left((a, c), (0.321e^{i2\pi(0.209)}, 0.372e^{i2\pi(0.371)}), (0.123e^{i2\pi(0.233)}, 0.373e^{i2\pi(0.345)})\right), \\ \left((a, l), (0.313e^{i2\pi(0.292)}, 0.435e^{i2\pi(0.381)}), (0.123e^{i2\pi(0.233)}, 0.374e^{i2\pi(0.329)})\right), \\ \left((b, a), (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.217)}), (0.123e^{i2\pi(0.233)}, 0.347e^{i2\pi(0.229)})\right), \\ \left((b, b), (0.317e^{i2\pi(0.211)}, 0.421e^{i2\pi(0.212)}), (0.346e^{i2\pi(0.333)}, 0.315e^{i2\pi(0.213)})\right), \\ \left((b, c), (0.317e^{i2\pi(0.209)}, 0.421e^{i2\pi(0.371)}), (0.346e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)})\right), \\ \left((b, l), (0.313e^{i2\pi(0.211)}, 0.435e^{i2\pi(0.381)}), (0.296e^{i2\pi(0.333)}, 0.374e^{i2\pi(0.329)})\right), \\ \left((c, a), (0.321e^{i2\pi(0.209)}, 0.372e^{i2\pi(0.371)}), (0.123e^{i2\pi(0.233)}, 0.373e^{i2\pi(0.345)})\right), \\ \left((c, b), (0.317e^{i2\pi(0.209)}, 0.421e^{i2\pi(0.371)}), (0.346e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)})\right), \\ \left((c, c), (0.348e^{i2\pi(0.209)}, 0.331e^{i2\pi(0.371)}), (0.369e^{i2\pi(0.301)}, 0.373e^{i2\pi(0.345)})\right), \\ \left((c, l), (0.313e^{i2\pi(0.209)}, 0.435e^{i2\pi(0.381)}), (0.296e^{i2\pi(0.301)}, 0.374e^{i2\pi(0.345)})\right), \\ \left((l, a), (0.313e^{i2\pi(0.292)}, 0.435e^{i2\pi(0.381)}), (0.123e^{i2\pi(0.233)}, 0.374e^{i2\pi(0.329)})\right), \\ \left((l, b), (0.313e^{i2\pi(0.211)}, 0.435e^{i2\pi(0.381)}), (0.296e^{i2\pi(0.333)}, 0.374e^{i2\pi(0.329)})\right), \\ \left((l, c), (0.313e^{i2\pi(0.209)}, 0.435e^{i2\pi(0.381)}), (0.296e^{i2\pi(0.301)}, 0.374e^{i2\pi(0.345)})\right), \\ \left((l, l), (0.313e^{i2\pi(0.323)}, 0.435e^{i2\pi(0.381)}), (0.296e^{i2\pi(0.357)}, 0.374e^{i2\pi(0.329)})\right) \end{array} \right\}$$

A CLD-partial order-FR $\bar{\mathcal{R}}$ is

$$\bar{\mathcal{R}} = \left\{ \begin{array}{l} \left((a,a),(0.321e^{i2\pi(0.292)},0.372e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\right), \\ \left((a,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.217)}),(0.123e^{i2\pi(0.233)},0.347e^{i2\pi(0.229)})\right), \\ \left((b,b),(0.317e^{i2\pi(0.211)},0.421e^{i2\pi(0.212)}),(0.346e^{i2\pi(0.333)},0.315e^{i2\pi(0.213)})\right), \\ \left((c,c),(0.348e^{i2\pi(0.209)},0.331e^{i2\pi(0.371)}),(0.369e^{i2\pi(0.301)},0.373e^{i2\pi(0.345)})\right), \\ \left((l,a),(0.313e^{i2\pi(0.292)},0.435e^{i2\pi(0.381)}),(0.123e^{i2\pi(0.233)},0.374e^{i2\pi(0.329)})\right), \\ \left((l,b),(0.313e^{i2\pi(0.211)},0.435e^{i2\pi(0.381)}),(0.296e^{i2\pi(0.333)},0.374e^{i2\pi(0.329)})\right), \\ \left((l,c),(0.313e^{i2\pi(0.209)},0.435e^{i2\pi(0.381)}),(0.296e^{i2\pi(0.301)},0.374e^{i2\pi(0.345)})\right), \\ \left((l,l),(0.313e^{i2\pi(0.323)},0.435e^{i2\pi(0.381)}),(0.296e^{i2\pi(0.357)},0.374e^{i2\pi(0.329)})\right) \end{array} \right\}$$

The Hasse Diagram for this relation is given in Figure 1.

Definition 15. Suppose a Hasse diagram that illustrate an CLD-partial order-FR, then the following element is referred to as:

i. The maximal element if there is no other element greater than it. e.g., an element a is maximal if there is no other element b such that $a \leq b$ and $a \neq b$.
ii. The minimal element if there is no other element smaller than it. e.g., an element a is minimal if there is no other element b such that $b \leq a$ and $a \neq b$.
iii. The maximum or the greatest element if all element associated with it are smaller than it. e.g., an element a is maximum if for every element b such that $b < a$.
iv. The minimum or the least element if all element associated with it are greater than it. e.g., an element a is minimum if for every element b such that $a < b$.

In Figure 1, the elements b and c are maximum elements, while l is the minimal as well as minimum element. The only maximal element is b.

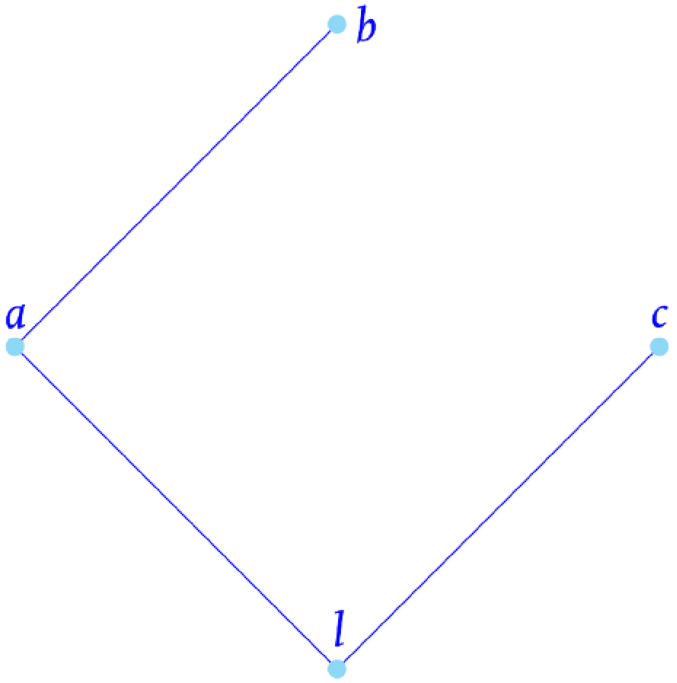

Fig. 1. Hasse Diagram for $\bar{\mathcal{R}}$ in example 15

Definition 16. If $\mathcal{W}$ represent any non-empty subset of an CLD-partial order-FS τ_f, then an element $r \in \bar{\mathcal{R}} \subseteq \tau_f \times \tau_f$ is referred to as:

i. Upper bound of $\mathcal{W}$ if $(b, a) \in \bar{\mathcal{R}}$, for all $b \in \mathcal{W}$.
ii. Lower bound of $\mathcal{W}$ if $(a, b) \in \bar{\mathcal{R}}$, for all $b \in \mathcal{W}$.
iii. Supremum of $\mathcal{W}$ if it is a least upper bound of $\mathcal{W}$.
iv. Infimum of $\mathcal{W}$ if it is a greatest lower bound of $\mathcal{W}$.

5. Application

This section presents the implementation of proposed work in application by applying examined relations and their various forms in data loss prevention technique of cybersecurity.

5.1 Data loss prevention (DLP)

A data breach or leak occurs when private, protected, or sensitive information is disclosed to an untrustworthy source. Insider trading by employees or former employees of a company, hacker assaults, or unintended data loss or exposure are all potential causes of data breaches.

Unauthorized duplication or transmission of data without affecting the original material is known as information leakage, also known as exfiltration. In certain cases, breaches result in the complete loss of data, such as ransomware attacks, in which hackers encrypt data to prevent the owner from accessing it.

Thus, a vital part of contemporary information security measures is data loss prevention (DLP), which protects sensitive data from illegal access, disclosure, or exfiltration. Through the implementation of policies and controls that regulate the usage, storage, and transport of private information inside an organization, DLP solutions are intended to identify, track, and avert data breaches.

Data loss prevention (DLP) is a critical component of cybersecurity solutions that protect sensitive data from unauthorized access, leakage, or theft. DLP solutions help businesses monitor, identify, and prevent the unauthorized transmission or theft of sensitive data across networks, endpoints, and cloud-based settings.

5.1.1 Security techniques of DLP

DLP uses following data security techniques:

i. Sensitive Data Discovery: DLP systems may scan and detect sensitive data on an organization's network, such as personally identifiable information (PII), financial data, intellectual property, and private documents. Organizations may improve their data landscape and security measures by categorizing and marking sensitive data.
ii. Data Surveillance and Inspection: DLP technologies continually monitor data in motion, at rest, and in use to keep track of how sensitive information is obtained, shared, and utilized inside the business. This surveillance allows for finding any illegal acts or legislative infractions in actual time.
iii. Policy Enforcement: DLP systems let firms develop and implement data security rules based on regulatory requirements, industry standards, and internal guidelines. Policies might include guidelines for data management, access restrictions, encryption, and data protection to help prevent data loss issues.
iv. Data Encryption: DLP systems frequently feature encrypted features to safeguard sensitive data in route and at rest. Encryption guarantees that data, even if intercepted or stolen, remains unintelligible and safe, lowering the risk of hacking and illegal access.

v. Endpoint Guarding: DLP systems may be installed on endpoints such as laptops, desktop computers, and mobile devices to monitor and regulate data transfers and prevent data leakage via removable storage devices, email attachments, or cloud services. Endpoint DLP protects data on gadgets used by employees both within and outside of the company's intranet.
vi. Cloud Data Protection: As cloud services become more widely used, DLP systems' capabilities expand to secure data stored in cloud settings. Organizations may use DLP policies to govern data access, sharing, and storage in cloud services, limiting the risk of information and compliance breaches.
vii. User Activity Monitoring: DLP systems monitor user behavior and activities involving highly confidential information access & usage. Organizations can detect dangers from insiders, illegal data transfers, and hazardous behaviors that might result in data loss events by monitoring how people interact with material.

Thus, through the use of content discovery, encryption, access restrictions, and monitoring systems, DLP solutions let businesses track sensitive data consumption, proactively detect sensitive information, and stop it from being disclosed without authorization.

In addition, the proliferation of mobile devices, cloud computing, and remote work settings, among other technological advancements, has increased the attack surface for possible data breaches. DLP systems have therefore evolved to meet these new difficulties by adding protection to networks, endpoints, and cloud environments, guaranteeing thorough data security on a variety of platforms.

Adopting strong DLP protections is essential for organizations, as the volume of data collected and exchanged across digital platforms has increased and so has the danger of data breaches. By taking these precautions, firms may preserve consumer trust, safeguard intellectual property, comply with legal obligations, and lessen the financial and reputational harm caused by data loss events.

Furthermore, by implementing these strategies and technologies, organizations can effectively secure data in any state, ensuring that sensitive information is protected against interception, unauthorized access, and data breaches. For instance, following securing data states are classified below:

i. Securing data in motion: A Data in motion in DLP is secured by utilizing network monitoring techniques, such as deep packet inspection and secure gateways, to identify and stop unwanted data transfers, as well as by encrypting data during transmission using technologies like TLS and VPNs.
ii. Securing data at rest: Implementing access controls to guarantee that only authorized users can access sensitive information and encrypting stored data to prevent unauthorized access are two ways to secure data at rest in DLP. In order to identify and address possible security risks, observe and audit data access on a regular basis.
iii. Securing data in use: In order to secure data used in DLP, use contextual policies to enforce access controls and stop unauthorized use or leakage, as well as real-time monitoring and analysis of data exchanges.
iv. Securing endpoints: Utilize device control mechanisms to limit the usage of external storage devices and unauthorized programs, and deploy endpoint protection software to monitor and regulate data activities in order to secure endpoints in DLP.
v. Data identification: Utilizing continuous learning approaches to adjust to changing data patterns and risks, secure data identification via machine learning in DLP involves training models with a variety of datasets to improve recognition accuracy of sensitive information.

vi. Data detection: Secure data loss detection in DLP can be achieved by utilizing machine learning algorithms for precise anomaly detection, carefully crafting policies to track data movement, and identifying sensitive information using sophisticated content inspection methods.

5.1.2 Threats

Although, data loss prevention system has their own set of threats and challenges, even though they are crucial for safeguarding sensitive data. So thus, for DLP to be implemented successfully, it is essential to comprehend these threats (Table 1).

The following are the main threats related with DLP:

i. Insider Leaks: Disgruntled workers, former workers who still have access credentials to private networks, and business partners are examples of insider risks. They may be driven by retaliation, money, or information that has economic value.

ii. Extrusion by attackers: Sensitive data is kept by organizations on a variety of gadgets, including desktop computers, servers, thumb drives, laptops, and portable hard drives. Any of these devices might be inadvertently misplaced by organization personnel or physically taken by an attacker, leading to a breach.

iii. Unintended Disclosure: Many data breaches result from the inadvertent disclosure of private information rather than from an assault. IT workers may inadvertently expose a critical internal server to the Internet, or employees may access sensitive data and store it in an insecure place.

Table 1
Details of threats

Causes	Notation	Membership	Non-Membership	Parameter A	Parameter B
Insider leakage	T_1	$0.721e^{i2\pi(0.643)}$	$0.292e^{i2\pi(0.287)}$	$0.753\mathrm{e}^{i2\pi(0.633)}$	$0.246e^{i2\pi(0.329)}$
Extrusion by attackers	T_2	$0.425e^{i2\pi(0.311)}$	$0.526e^{i2\pi(0.412)}$	$0.546e^{i2\pi(0.523)}$	$0.415e^{i2\pi(0.413)}$
Unintended disclosure	T_3	$0.278e^{i2\pi(0.409)}$	$0.861e^{i2\pi(0.376)}$	$0.319e^{i2\pi(0.501)}$	$0.673e^{i2\pi(0.445)}$

5.1.3 Component of Data Loss Solution or States of securing data loss

Following main data loss states are supported by DLP systems to secure organizations from data breaches or leakage (Table 2):

i. Securing data in motion (transferred data)
ii. Securing endpoints
iii. Securing data-at-rest
iv. Securing data in use
v. Data identification
vi. Data leak detection

Table 2
Data loss states

Securing states	Notation	Membership	Non-Membership	Parameter A	Parameter B
Securing data in motion	S_m	$0.812e^{i2\pi(0.222)}$	$0.261e^{i2\pi(0.532)}$	$0.625e^{i2\pi(0.357)}$	$0.274e^{i2\pi(0.566)}$
Securing endpoints	S_e	$0.441e^{i2\pi(0.334)}$	$0.357e^{i2\pi(0.349)}$	$0.512e^{i2\pi(0.353)}$	$0.407e^{i2\pi(0.459)}$
Securing data at rest	S_r	$0.542e^{i2\pi(0.549)}$	$0.364e^{i2\pi(0.492)}$	$0.437e^{i2\pi(0.264)}$	$0.521e^{i2\pi(0.573)}$
Securing data in use	S_u	$0.642e^{i2\pi(0.359)}$	$0.327e^{i2\pi(0.492)}$	$0.671e^{i2\pi(0.257)}$	$0.332e^{i2\pi(0.595)}$
Data identification	S_i	$0.347e^{i2\pi(0.649)}$	$0.417e^{i2\pi(0.491)}$	$0.327e^{i2\pi(0.587)}$	$0.599e^{i2\pi(0.345)}$
Data leak detection	S_d	$0.243e^{i2\pi(0.319)}$	$0.558e^{i2\pi(0.482)}$	$0.415e^{i2\pi(0.457)}$	$0.518e^{i2\pi(0.397)}$

5.2 Calculations

In this section, the relationships are examined, focusing on efficacy and inefficacy of each cybersecurity technique in preventing data loss. We perform the following analysis. The following two CLDFSs ν_f and τ_f, illustrating the set of components of data loss solution and the set of threads are carried out, respectively.

$$\nu_f=\left\{\begin{matrix}\begin{pmatrix}S_m,(0.812e^{i2\pi(0.222)},0.261e^{i2\pi(0.532)}),\\(0.625e^{i2\pi(0.357)},0.274e^{i2\pi(0.566)})\end{pmatrix},\begin{pmatrix}S_e,(0.441e^{i2\pi(0.334)},0.357e^{i2\pi(0.349)}),\\(0.512e^{i2\pi(0.353)},0.407e^{i2\pi(0.459)})\end{pmatrix},\\\begin{pmatrix}S_r,(0.542e^{i2\pi(0.549)},0.364e^{i2\pi(0.492)}),\\(0.437e^{i2\pi(0.264)},0.521e^{i2\pi(0.573)})\end{pmatrix},\begin{pmatrix}S_u,(0.642e^{i2\pi(0.359)},0.327e^{i2\pi(0.492)}),\\(0.671e^{i2\pi(0.257)},0.332e^{i2\pi(0.595)})\end{pmatrix},\\\begin{pmatrix}S_i,(0.347e^{i2\pi(0.649)},0.417e^{i2\pi(0.491)}),\\(0.327e^{i2\pi(0.587)},0.599e^{i2\pi(0.345)})\end{pmatrix},\begin{pmatrix}S_d,(0.243e^{i2\pi(0.319)},0.558e^{i2\pi(0.482)}),\\(0.415e^{i2\pi(0.457)},0.518e^{i2\pi(0.397)})\end{pmatrix}\end{matrix}\right\}$$

$$\tau_f=\left\{\begin{matrix}\begin{pmatrix}T_1,(0.821e^{i2\pi(0.643)},0.292e^{i2\pi(0.287)}),\\(0.753e^{i2\pi(0.633)},0.246e^{i2\pi(0.329)})\end{pmatrix},\begin{pmatrix}T_2,(0.525e^{i2\pi(0.311)},0.326e^{i2\pi(0.412)}),\\(0.546e^{i2\pi(0.523)},0.415e^{i2\pi(0.413)})\end{pmatrix},\\\begin{pmatrix}T_3,(0.378e^{i2\pi(0.409)},0.461e^{i2\pi(0.376)}),\\(0.319e^{i2\pi(0.501)},0.673e^{i2\pi(0.445)})\end{pmatrix}\end{matrix}\right\}$$

Thus, we utilize the cartesian product to determine the effectiveness of specific threats against specific components of data loss solution. Which is given below:

$$\nu_f\times\tau_f=\left\{\begin{matrix}\left((S_m,T_1),(0.721e^{i2\pi(0.222)},0.292e^{i2\pi(0.532)}),(0.625e^{i2\pi(0.357)},0.246e^{i2\pi(0.566)})\right),\\\left((S_m,T_2),(0.425e^{i2\pi(0.222)},0.326e^{i2\pi(0.217)}),(0.512e^{i2\pi(0.353)},0.415e^{i2\pi(0.566)})\right),\\\left((S_m,T_3),(0.278e^{i2\pi(0.222)},0.461e^{i2\pi(0.532)}),(0.319e^{i2\pi(0.357)},0.673e^{i2\pi(0.566)})\right),\\\left((S_e,T_1),(0.441e^{i2\pi(0.334)},0.357e^{i2\pi(0.349)}),(0.512e^{i2\pi(0.353)},0.407e^{i2\pi(0.459)})\right),\\\left((S_e,T_2),(0.425e^{i2\pi(0.311)},0.357e^{i2\pi(0.412)}),(0.512e^{i2\pi(0.353)},0.415e^{i2\pi(0.459)})\right),\\\left((S_e,T_3),(0.278e^{i2\pi(0.334)},0.461e^{i2\pi(0.376)}),(0.319e^{i2\pi(0.353)},0.673e^{i2\pi(0.459)})\right),\\\left((S_r,T_1),(0.542e^{i2\pi(0.549)},0.364e^{i2\pi(0.492)}),(0.437e^{i2\pi(0.264)},0.521e^{i2\pi(0.573)})\right),\\\left((S_r,T_2),(0.425e^{i2\pi(0.311)},0.364e^{i2\pi(0.492)}),(0.437e^{i2\pi(0.264)},0.521e^{i2\pi(0.573)})\right),\\\left((S_r,T_3),(0.278e^{i2\pi(0.409)},0.461e^{i2\pi(0.492)}),(0.319e^{i2\pi(0.264)},0.673e^{i2\pi(0.573)})\right),\\\left((S_u,T_1),(0.642e^{i2\pi(0.359)},0.327e^{i2\pi(0.412)}),(0.671e^{i2\pi(0.257)},0.332e^{i2\pi(0.595)})\right),\\\left((S_u,T_2),(0.425e^{i2\pi(0.311)},0.327e^{i2\pi(0.492)}),(0.546e^{i2\pi(0.257)},0.415e^{i2\pi(0.595)})\right),\\\left((S_u,T_3),(0.278e^{i2\pi(0.359)},0.461e^{i2\pi(0.492)}),(0.319e^{i2\pi(0.257)},0.673e^{i2\pi(0.595)})\right),\\\left((S_i,T_1),(0.347e^{i2\pi(0.643)},0.417e^{i2\pi(0.491)}),(0.327e^{i2\pi(0.587)},0.599e^{i2\pi(0.345)})\right),\\\left((S_i,T_2),(0.347e^{i2\pi(0.311)},0.417e^{i2\pi(0.491)}),(0.327e^{i2\pi(0.523)},0.599e^{i2\pi(0.413)})\right),\\\left((S_i,T_3),(0.243e^{i2\pi(0.319)},0.461e^{i2\pi(0.491)}),(0.319e^{i2\pi(0.501)},0.673e^{i2\pi(0.445)})\right),\\\left((S_d,T_1),(0.243e^{i2\pi(0.319)},0.558e^{i2\pi(0.482)}),(0.415e^{i2\pi(0.457)},0.518e^{i2\pi(0.397)})\right),\\\left((S_d,T_2),(0.243e^{i2\pi(0.319)},0.558e^{i2\pi(0.482)}),(0.415e^{i2\pi(0.457)},0.518e^{i2\pi(0.413)})\right),\\\left((S_d,T_3),(0.243e^{i2\pi(0.319)},0.558e^{i2\pi(0.482)}),(0.319e^{i2\pi(0.457)},0.673e^{i2\pi(0.445)})\right)\end{matrix}\right\}$$

The $\nu_f \times \tau_f$ represents the relation between the set of components of data loss solution and the set of threats. For an ordered pair $\begin{pmatrix} (\mathrm{a},\mathrm{b}), \left(\mathcal{A}_{(\tau_f\times\nu_f)\mathfrak{M}}(\mathrm{a},\mathrm{b})e^{i2\left(\mathcal{P}_{(\tau_f\times\nu_f)\mathfrak{M}}(\mathrm{a},\mathrm{b})\right)\pi}, \mathcal{A}_{(\tau_f\times\nu_f)\mathfrak{N}}(\mathrm{a},\mathrm{b})e^{i2\left(\mathcal{P}_{(\tau_f\times\nu_f)\mathfrak{N}}(\mathrm{a},\mathrm{b})\right)\pi}\right), \\ \left(\mathcal{A}_{(\tau_f\times\nu_f)\mathfrak{A}}, e^{i2\left(\mathcal{P}_{(\tau_f\times\nu_f)\mathfrak{A}}\right)\pi}, \mathcal{A}_{(\tau_f\times\nu_f)\mathfrak{B}}e^{i2\left(\mathcal{P}_{(\tau_f\times\nu_f)\mathfrak{B}}\right)\pi}\right) \end{pmatrix}$, its specifically illustrates how the first element affects or influences the second within the ordered pair. Whereas the membership degree signifies the potency of security of data in different states against certain threats to address a particular time-related risk. However, the non-membership degree signifies the consequences related to threats over the securing data states. Moreover, the parameters related to membership and non-membership degrees represent the requirements of securing data states to be confronted over the penetration through threats to ensure data security. Such as, $\left((S_m, T_1), (0.721e^{i2\pi(0.222)}, 0.292e^{i2\pi(0.532)}), (0.625e^{i2\pi(0.357)}, 0.246e^{i2\pi(0.566)})\right)$ being an ordered pair illustrates that a data in motion can be secured more potently against the insider leakage risk. The values of degrees are stated as: the value of membership degree clarifies that security potency of data in motion against the exposure of data through insider leakage is 72.1% over the time span of approximately one-fourth unit and the value of non-membership degree signifies that the possibility of penetration of securing data in motion through insider leakage is 29.2% over the time span of approximately half unit. whereas, the values of parameters represent that the security of data in motion is required to be 62.5% in time span of approximately one-third to tackle the risk of insider leakage whereas the parameter value of non-membership indicates that 24.6% risk can be tolerated in time span of approximately half unit.

6. Comparative analysis

This section compares intended CLDFR structure to current structures, such as CFRs, CIFRs, CPyFRs, CqROFRs and LDFRs, with the aim of suggested framework stability authentication.

6.1 Comparison with FRs and CFRs:

In FRs, the focus is only on real-valued degree of membership, while in CFRs a real-imaginary valued membership grades in term of amplitude and phase term are utilized concentrating on relationship grades without any limitations. But in CLDFRs a real-imaginary valued membership grade as well as non-membership grade along with parameters are introduced focusing on the ability to evaluate any relationship's both potency and impotency.

Moreover, in order to precisely model multivariable problems, the FRs and CFRs fail. So, the only way to handle such problems is through CLDFRs.

Since CFR's is the extended structure of FRs structure as discussed, so thus a thorough comparison of CFR's and CLDFRs is presented below.

By using CFR's and considering the following two CFs ν_f and τ_f, illustrating the set of components of data loss solution and the set of threats respectively, we analyze the problem stated in section 4.2. In order to simplify the computation process and finalize the comparative analysis certain components of data loss solution and threats sources are excluded.

$$\nu_f = \begin{Bmatrix} (S_m, (0.812e^{i2\pi(0.222)})), (S_r, (0.542e^{i2\pi(0.549)})), \\ (S_u, (0.642e^{i2\pi(0.359)})), (S_e, (0.441e^{i2\pi(0.334)})) \end{Bmatrix}$$

$$\tau_f = \left\{\left(T_1, (0.821e^{i2\pi(0.643)})\right), \left(T_2, (0.525e^{i2\pi(0.311)})\right)\right\}$$

The CFR $\bar{\mathcal{R}}$ between ν_f and τ_f

$$\bar{\mathcal{R}} = \left\{ \begin{array}{c} \left((S_m, T_1), (0.721e^{i2\pi(0.222)})\right), \left((S_m, T_2), (0.425e^{i2\pi(0.222)})\right), \\ \left((S_e, T_1), (0.441e^{i2\pi(0.334)})\right), \left((S_e, T_2), (0.425e^{i2\pi(0.311)})\right), \\ \left((S_r, T_1), (0.542e^{i2\pi(0.549)})\right), \left((S_r, T_2), (0.425e^{i2\pi(0.311)}\right), \\ \left((S_u, T_1), (0.642e^{i2\pi(0.359)})\right), \left((S_u, T_2), (0.425e^{i2\pi(0.311)})\right) \end{array} \right\}$$

As observed from above, CFR $\bar{\mathcal{R}}$ simply provide information regarding the membership grade. That's, because CFRs structure lacks any degree of non-membership, it only reveals the efficacy of security of data in different states against certain threats to address a particular time-related risk and hides the consequences related to threats over the securing data states. As a result, these structures provide limited amount of information and have significant restrictions.

6.2 Comparison with IFRs, CIFRs, PyFRs, and CPyFRs

As CLDFR, being a combination of real-imaginary valued membership degree, non-membership degree along with parameters in terms of amplitude and phase term model uncertainty with a structured, constraint-based approach. But, IFRs limited to real numbers, are restricted to handle single-variable problems. As a result, they lack a capability to address the problems related to time variations or those with phase transitions. Whereas, CIFRs structure deal with the real-imaginary valued membership degree and non-membership degree having amplitude and phase term but they are less effective due to the absence of parameters. Also, the sum of amplitude terms and phase terms of membership degree and non-membership degree respectively, must lies in unit interval, making CFRs restricted to limitations.

Furthermore, PyFRs and CPyFRs, also have some limitations regarding membership and non-membership grades and are incapable of dealing with parameters. so, it becomes less effective to handle problems more precisely.

Thus, an advantage of CLDFRs over CIFRs and CPyFRs lies in their structured approach having parameters and negligence of restrictions related to degrees.

6.3 Comparison with LDFRs

The shortcoming of IFSs, PyFSs and q-ROFSs concerning membership and non-membership grades, as well as their incapacity to manage parameterization, are addressed by LDFSs. LDFRs broadens the analytical space by removing these limitations. By adding reference factor, LDFRs improve on current approaches by enabling the free selection of membership and non-membership grades. But LDFRs, in which the grades are limited to real valued, are unable to handle problems involving phase terms or time span. Thus, CLDFRs, which covers the problems that include both real and imaginary parts in term of amplitude and phase term, extend LDFRs by incorporating real-imaginary valued membership, non-membership grades and parameters grades.

By using LDFRs and considering the following two CFs ν_f and τ_f, illustrating the set of components of data loss solution and the set of threats respectively, we analyze the problem stated in section 4.2. In order to simplify the computation process and finalize the comparative analysis, certain components of data loss solution and threats sources are excluded.

$$\nu_f = \left\{ \begin{pmatrix} S_m, (0.812, 0.261), \\ (0.625, 0.274) \end{pmatrix}, \begin{pmatrix} S_e, (0.441, 0.357), \\ (0.512, 0.407) \end{pmatrix}, \begin{pmatrix} S_r, (0.542, 0.364), \\ (0.437, 0.521) \end{pmatrix}, \begin{pmatrix} S_u, (0.642, 0.327), \\ (0.671, 0.332) \end{pmatrix} \right\}$$

$$\tau_f = \left\{ \begin{pmatrix} T_1, (0.821, 0.292), \\ (0.753, 0.246) \end{pmatrix}, \begin{pmatrix} T_2, (0.525, 0.326), \\ (0.546, 0.415) \end{pmatrix} \right\}$$

The LDFR $\bar{\mathcal{R}}$ between ν_f and τ_f

$$\nu_f \times \tau_f = \left\{ \begin{array}{l} \big((S_m,T_1),(0.721,0.292),(0.625,0.246)\big),\big((S_m,T_2),(0.425,0.326),(0.512,0.415)\big), \\ \big((S_e,T_1),(0.441,0.357),(0.512,0.407)\big),\big((S_e,T_2),(0.425,0.357),(0.512,0.415)\big), \\ \big((S_r,T_1),(0.542,0.364),(0.437,0.521)\big),\big((S_r,T_2),(0.425,0.364),(0.437,0.521)\big), \\ \big((S_u,T_1),(0.642,0.327),(0.671,0.332)\big),\big((S_u,T_2),(0.425,0.327),(0.546,0.415)\big), \end{array} \right\}$$

As shown above, LDFR focus only real-valued problems, simply providing information based on real valued membership, non-membership grade and parameters. it only reveals the efficacy of security of data in different states against certain threats to address risk without being related to time and hides the consequences related to threats over the securing data states.

7. Conclusion

In this paper, two new concepts are introduced: the cartesian product of two complex linear Diophantine fuzzy sets CLDFSs and the complex linear Diophantine fuzzy relation (CLDFR). Furthermore, the complex linear Diophantine equivalence fuzzy relation (CLD- equivalence-FR), CLD-partial-FR, CLD- total order-FR, CLD- composite-FR and many other forms of CLDFRs are also defined. Moreover, for the CLD- partial-FR and CLD- partial-FS, the Hasse diagram has been presented. In addition, definitions of terms and ideas associated with Hasse diagram have also been provided. For every definition, suitable examples are provided, and various results are demonstrated for different types of CLDFRs. Furthermore, the suggested concepts are applied to examine the relation between the components of data loss solution and risk related to threats. Through a contrast comparison with different substitute mathematical methodologies, the section named as comparative analysis demonstrate the superiority CLDFRs. Furthermore, it also summarizes the extensive framework of CLDFRs as well as the shortcomings of earlier frameworks.

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Klir, G. J., & Folger, T. A. (1988). Fuzzy sets, uncertainty, and information. Prentice Hall.

[3] Mendel, J. M. (1995). Fuzzy logic systems for engineering: A tutorial. Proceedings of the IEEE, 83(3), 345–377. https://doi.org/10.1109/5.364485

[4] Goguen, J. A. (1974). Concept representation in natural and artificial languages: Axioms, extensions and applications for fuzzy sets. International Journal of Man-Machine Studies, 6(5), 513–561.

[5] Żywica, P. (2018). Modelling medical uncertainties with use of fuzzy sets and their extensions. In Information Processing and Management of Uncertainty in Knowledge-Based Systems. Applications: 17th International Conference, IPMU 2018, Cádiz, Spain, June 11-15, 2018, Proceedings, Part III 17 (pp. 369-380). Springer International Publishing.

[6] Dubois, D., & Prade, H. (2012). Gradualness, uncertainty and bipolarity: Making sense of fuzzy sets. Fuzzy Sets and Systems, 192, 3–24. https://doi.org/10.1016/j.fss.2010.11.007

[7] Ramot, D., Milo, R., Friedman, M., & Kandel, A. (2002). Complex fuzzy sets. IEEE Transactions on Fuzzy Systems, 10(2), 171–186. https://doi.org/10.1109/91.995119

[8] Yazdanbakhsh, O., & Dick, S. (2018). A systematic review of complex fuzzy sets and logic. Fuzzy Sets and Systems, 338, 1–22. https://doi.org/10.1016/j.fss.2017.01.010

[9] Liu, P., Ali, Z., & Mahmood, T. (2020). The distance measures and cross-entropy based on complex fuzzy sets and their application in decision making. Journal of Intelligent & Fuzzy Systems, 39(3), 3351–3374. https://doi.org/10.3233/JIFS-191718

[10] Sobhi, S., & Dick, S. (2023). An investigation of complex fuzzy sets for large-scale learning. Fuzzy Sets and Systems, 471, 108660. https://doi.org/10.1016/j.fss.2023.108660

[11] Tamir, D. E., Rishe, N. D., & Kandel, A. (2015). Complex fuzzy sets and complex fuzzy logic: An overview of theory and applications. In Fifty years of fuzzy logic and its applications (pp. 661–681). Springer. https://doi.org/10.1007/978-3-319-19683-1_31

[12] Atanassov, K. T. (1999). Intuitionistic fuzzy sets. Physica-Verlag.

[13] Burillo, P., & Bustince, H. (1995). Intuitionistic fuzzy relations (Part I). Mathware and Soft Computing, 2(1), 5–38.

[14] Alkouri, A. M. D. J. S., & Salleh, A. R. (2012). Complex intuitionistic fuzzy sets. AIP Conference Proceedings, 1482(1), 464–470. https://doi.org/10.1063/1.4757515

[15] Rahman, A. U., Ahmad, M. R., Saeed, M., Ahsan, M., Arshad, M., & Ihsan, M. (2020). A study on fundamentals of refined intuitionistic fuzzy set with some properties. Journal of Fuzzy Extension and Applications, 1(4), 279–292. https://doi.org/10.22105/jfea.2020.261946.1067

[16] Yu, D., Sheng, L., & Xu, Z. (2022). Analysis of evolutionary process in intuitionistic fuzzy set theory: A dynamic perspective. Information Sciences, 601, 175–188. https://doi.org/10.1016/j.ins.2022.04.019

[17] De, S. K., Biswas, R., & Roy, A. R. (2001). An application of intuitionistic fuzzy sets in medical diagnosis. Fuzzy Sets and Systems, 117(2), 209–213. https://doi.org/10.1016/S0165-0114(98)00235-8

[18] Liu, P. (2013). Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. IEEE Transactions on Fuzzy Systems, 22(1), 83–97. https://doi.org/10.1109/TFUZZ.2013.2248736

[19] Vlachos, I. K., & Sergiadis, G. D. (2007). Intuitionistic fuzzy information–Applications to pattern recognition. Pattern Recognition Letters, 28(2), 197–206. https://doi.org/10.1016/j.patrec.2006.07.004

[20] Li, D. F. (2005). Multiattribute decision making models and methods using intuitionistic fuzzy sets. Journal of Computer and System Sciences, 70(1), 73–85. https://doi.org/10.1016/j.jcss.2004.06.002

[21] Nasir, A., Jan, N., Gumaei, A., Khan, S. U., & Al-Rakhami, M. (2021). Evaluation of the economic relationships on the basis of statistical decision-making in complex neutrosophic environment. Complexity, 2021, 5595474. https://doi.org/10.1155/2021/5595474

[22] Nasir, A., Jan, N., Gumaei, A., Khan, S. U., & Albogamy, F. R. (2021). Cybersecurity against the loopholes in industrial control systems using interval-valued complex intuitionistic fuzzy relations. Applied Sciences, 11(16), 7668. https://doi.org/10.3390/app11167668

[23] Garg, H., & Rani, D. (2020). Novel aggregation operators and ranking method for complex intuitionistic fuzzy sets and their applications to decision-making process. Artificial Intelligence Review, 53, 3595–3620. https://doi.org/10.1007/s10462-019-09772-x

[24] Yager, R. R. (2013). Pythagorean membership grades in multicriteria decision making. IEEE Transactions on Fuzzy Systems, 22(4), 958–965. https://doi.org/10.1109/TFUZZ.2013.2278989

[25] Yager, R. R. (2013). Pythagorean fuzzy subsets. In 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS) (pp. 57–61). IEEE. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

[26] Ullah, K., Mahmood, T., Ali, Z., & Jan, N. (2020). On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. Complex & Intelligent Systems, 6, 15–27. https://doi.org/10.1007/s40747-019-0103-6

[27] Peng, X., & Selvachandran, G. (2019). Pythagorean fuzzy set: State of the art and future directions. Artificial Intelligence Review, 52, 1873–1927. https://doi.org/10.1007/s10462-017-9596-9

[28] Saikia, B., Dutta, P., & Talukdar, P. (2023). An advanced similarity measure for Pythagorean fuzzy sets and its applications in transportation problem. Artificial Intelligence Review, 56(11), 12689–12724. https://doi.org/10.1007/s10462-023-10421-7

[29] Khan, M. J., Alcantud, J. C. R., Kumam, W., Kumam, P., & Alreshidi, N. A. (2023). Expanding Pythagorean fuzzy sets with distinctive radii: Disc Pythagorean fuzzy sets. Complex & Intelligent Systems, 9(6), 7037–7054. https://doi.org/10.1007/s40747-023-01062-y

[30] Pan, L., Deng, Y., & Cheong, K. H. (2023). Quaternion model of Pythagorean fuzzy sets and its distance measure. Expert Systems with Applications, 213, 119222. https://doi.org/10.1016/j.eswa.2022.119222

[31] Labassi, F., Ur Rehman, U., Alsuraiheed, T., Mahmood, T., & Khan, M. A. (2024). A novel approach towards complex Pythagorean fuzzy sets and their applications in visualization technology. IEEE Access, 12, 12345–12360. https://doi.org/10.1109/ACCESS.2024.3393138

[32] Akram, M., Zahid, K., & Kahraman, C. (2023). New optimization technique for group decision analysis with complex Pythagorean fuzzy sets. Journal of Intelligent & Fuzzy Systems, 44(3), 3621–3645.

[33] Yager, R. R. (2016). Generalized orthopair fuzzy sets. IEEE Transactions on Fuzzy Systems, 25(5), 1222–1230. https://doi.org/10.1109/TFUZZ.2016.2604005

[34] Liu, P., Mahmood, T., & Ali, Z. (2019). Complex q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making. Information, 11(1), 5. https://doi.org/10.3390/info11010005

[35] Garg, H., Gwak, J., Mahmood, T., & Ali, Z. (2020). Power aggregation operators and VIKOR methods for complex q-rung orthopair fuzzy sets and their applications. Mathematics, 8(4), 538. https://doi.org/10.3390/math8040538

[36] Peng, X., & Luo, Z. (2021). A review of q-rung orthopair fuzzy information: Bibliometrics and future directions. Artificial Intelligence Review, 54, 3361–3430. https://doi.org/10.1007/s10462-020-09926-2

[37] Khan, M. J., Kumam, P., & Shutaywi, M. (2021). Knowledge measure for the q-rung orthopair fuzzy sets. International Journal of Intelligent Systems, 36(2), 628–655. https://doi.org/10.1002/int.22313

[38] Demir Uslu, Y., Dinçer, H., Yüksel, S., Gedikli, E., & Yılmaz, E. (2022). An integrated decision-making approach based on q-rung orthopair fuzzy sets in service industry. International Journal of Computational Intelligence Systems, 15(1), 14. https://doi.org/10.1007/s44196-022-00069-6

[39] Akram, M., Naz, S., & Ziaa, F. (2023). Novel decision-making framework based on complex q-rung orthopair fuzzy information. Scientia Iranica, 30(4), 1450–1479. https://doi.org/10.24200/sci.2021.55413.4209

[40] Javeed, S., Javed, M., Shafique, I., Shoaib, M., Khan, M. S., Cui, L., & Alshamrani, A. M. (2024). Complex q-rung orthopair fuzzy Yager aggregation operators and their application to evaluate the best medical manufacturer. Applied Soft Computing, 157, 111234. https://doi.org/10.1016/j.asoc.2024.111532

[41] Riaz, M., & Hashmi, M. R. (2019). Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems. Journal of Intelligent & Fuzzy Systems, 37(4), 5417–5439. https://doi.org/10.3233/JIFS-190550

[42] Kamacı, H. (2022). Complex linear Diophantine fuzzy sets and their cosine similarity measures with applications. Complex & Intelligent Systems, 8(2), 1281–1305. https://doi.org/10.1007/s40747-021-00573-w

[43] Ayub, S., Gul, R., Shabir, M., Hummdi, A. Y., Aljaedi, A., & Bassfer, Z. (2024). A novel approach to roughness of linear Diophantine fuzzy sets by fuzzy relations and its application towards multi-criteria group decision-making. IEEE Access, 12, 12345–12360. https://doi.org/10.1109/ACCESS.2024.3386581

[44] Ayub, S., Shabir, M., Riaz, M., Aslam, M., & Chinram, R. (2021). Linear Diophantine fuzzy relations and their algebraic properties with decision making. Symmetry, 13(6), 945. https://doi.org/10.3390/sym13060945

[45] Zia, M. D., Yousafzai, F., Abdullah, S., & Hila, K. (2024). Complex linear Diophantine fuzzy sets and their applications in multi-attribute decision making. Engineering Applications of Artificial Intelligence, 132, 107953. https://doi.org/10.1016/j.engappai.2024.107953

[46] Guan, H., Yousafzai, F., Zia, M. D., Khan, M. U. I., Irfan, M., & Hila, K. (2022). Complex linear Diophantine fuzzy sets over AG-groupoids with applications in civil engineering. Symmetry, 15(1), 74. https://doi.org/10.3390/sym15010074