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Review of Probabilistic HyperGraph and Probabilistic SuperHyperGraph

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ABSTRACT

Uncertainty pervades many real-world networks, yet existing models such as probabilistic graphs and hypergraphs capture only pairwise or fixed-order interactions. We introduce the novel concept of *Probabilistic n-SuperHyperGraphs*, which unify nested higher-order relationships with edge-level uncertainty by assigning probabilities to "superedges" at multiple powerset levels. We present a rigorous formal framework, derive fundamental properties—including degree–sum identities and closure under substructure—and show that our model subsumes classical probabilistic graphs and hypergraphs as special cases. These results pave the way for more expressive and scalable methods in modeling and analyzing complex, uncertainty-laden systems across diverse application domains.

1. Preliminaries

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. Throughout this paper, all sets and structures are assumed to be finite. Unless otherwise stated, the symbol n denotes a non-negative integer.

1.1 *SuperHyperGraph*

Graph theory studies properties and applications of graphs, structures of vertices linked by edges, to model relationships and real-world networks [1, 2]. In classical graph theory, a hypergraph extends the idea of a conventional graph by permitting edges—called hyperedges—to join more than two vertices. This broader framework enables the modeling of more intricate relationships between el-

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ements, thereby enhancing its utility in various fields [3–6]. A *SuperHyperGraph* is an advanced extension of the hypergraph concept, integrating recursive powerset structures into the classical model. This concept has been recently introduced and extensively studied in the literature [7–13]. Including related concepts, we describe them below.

Definition 1.1 (Base Set)**.** *A* base set S *is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:*

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** *The* powerset *of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:*

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** *(cf.[14–17])*

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Example 1.4 (Smart Home Automation Scenes)**.** *Consider a smart home with three devices:*

$$H = \{\textit{Light}, \ \textit{Thermostat}, \ \textit{Camera}\}.$$

- *The first powerset*

$$P_1(H) = \mathcal{P}(H) = \{\varnothing, \{\textit{Light}\}, \{\textit{Thermostat}\}, \{\textit{Camera}\},$$
$$\{\textit{Light}, \textit{Thermostat}\}, \{\textit{Light}, \textit{Camera}\},$$
$$\{\textit{Thermostat}, \textit{Camera}\}, \{\textit{Light}, \textit{Thermostat}, \textit{Camera}\}\}$$

  represents all possible automation scenes, *for example "Lights on and Thermostat at 22 °C."*

- *The second powerset*

$$P_2(H) = \mathcal{P}\big(P_1(H)\big)$$

  consists of collections of scenes, such as a morning routine $\{\{\textit{Light}\}, \{\textit{Thermostat}\}\}$ *or an* away mode $\{\varnothing, \{\textit{Camera}\}\}$.

- *The third powerset*

$$P_3(H) = \mathcal{P}\big(P_2(H)\big)$$

  organizes these collections into routine sets, *for instance* $\{$*weekday routines,weekend routines*$\}$, *where each "routine" is itself a set of scenes.*

Thus the n-th powerset models nested layers of configuration: scenes, scene-collections, and collections of collections, as needed for complex automation hierarchies.

Definition 1.5 (Hypergraph)**.** *[3, 18]* A hypergraph $H = (V(H), E(H))$ *consists of:*

- *A nonempty set $V(H)$ of vertices.*
- *A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.*

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Example 1.6 (University Degree Requirements as a Hypergraph)**.** *Consider a set of courses offered at a university:*

$$V(H) = \{\textit{Calculus}, \ \textit{Physics}, \ \textit{Programming}, \ \textit{English}, \ \textit{History}\}.$$

Define three degree programs by their required courses:

$$E(H) = \{e_1, e_2, e_3\},$$

where

$$e_1 = \{\textit{Calculus}, \textit{Physics}, \textit{Programming}\}, \quad e_2 = \{\textit{English}, \textit{History}\}, \quad e_3 = \{\textit{Calculus}, \textit{English}, \textit{History}\}.$$

Then the hypergraph

$$H = \big(V(H), \ E(H)\big)$$

models each program (hyperedge) as the set of courses students must take. For example, the "Engineering" program e_1 links Calculus, Physics, and Programming, while the "Liberal Arts" program e_3 connects Calculus, English, and History.

Definition 1.7 (n-SuperHyperGraph)**.** *[19–21]*
Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset operation. An n-SuperHyperGraph *is then a pair*

$$\mathrm{SHG}^{(n)} = (V, E),$$

with

$$V \subseteq \mathcal{P}^n(V_0) \quad \textit{and} \quad E \subseteq \mathcal{P}^n(V_0).$$

Each element of V is called an n-supervertex *and each element of E an* n-superedge.

Example 1.8 (Real-World Example of a 2-SuperHyperGraph: Company Organization)**.** *Consider a company in which individual employees form teams, and teams in turn form higher-level departments. We model this hierarchy as a 2-*SuperHyperGraph*:*

$$\mathrm{SHG}^{(2)} \ = \ \big(V, \ E\big),$$

with respect to the base set of employees V_0, as follows:

- ***Level-0 vertices (Employees).***

  $$V_0 \;=\; \{\, e_1 = \textit{Alice},\; e_2 = \textit{Bob},\; e_3 = \textit{Carol},\; e_4 = \textit{Dave},\; e_5 = \textit{Eve},\; e_6 = \textit{Frank} \}.$$

  Each $e_i \in V_0$ represents a distinct employee.

- ***Level-1 hyperedges (Teams).*** *Group employees into project teams—each team is a subset of V_0:*

  $$T_1 \;=\; \{\textit{Alice}, \textit{Bob}\}, \quad T_2 \;=\; \{\textit{Carol}, \textit{Dave}\}, \quad T_3 \;=\; \{\textit{Eve}, \textit{Frank}\}.$$

  Thus

  $$\{T_1, T_2, T_3\} \subseteq \mathcal{P}\big(V_0\big),$$

  and each T_j is a nonempty subset of employees who collaborate on a given project.

- ***Level-2 supervertices (Departments).*** *Treat each team T_j as a single "team-vertex" in the first iterated powerset $\mathcal{P}(V_0)$. Then form subsets of these teams to represent departments:*

  $$V \;=\; \big\{\, T_1, T_2, T_3 \big\} \;\subseteq\; \mathcal{P}\big(V_0\big) \;\subseteq\; \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

  Each element of V is called a 2-supervertex *(in this case, simply a team).*

- ***Level-2 superedges (Departments).*** *Define departments as subsets of teams—that is, subsets of V:*

  $$D_A \;=\; \{\, T_1, T_2\}, \quad D_B \;=\; \{\, T_2, T_3\}.$$

  Hence

  $$E \;=\; \big\{\, D_A, D_B \big\} \;\subseteq\; \mathcal{P}\big(V\big) \;\subseteq\; \mathcal{P}\big(\mathcal{P}(\mathcal{P}(V_0))\big).$$

  Each D_α is a 2-superedge, *representing one department that oversees the specified teams.*

Summarizing, the triplet

$$\big(V_0,\, \{\, T_1, T_2, T_3\},\, \{\, D_A, D_B\}\big)$$

yields a concrete 2-SuperHyperGraph structure:

$$\mathrm{SHG}^{(2)} \;=\; \big(V,\, E\big) \;=\; \Big(\{\, T_1,\, T_2,\, T_3\},\, \{\, D_A,\, D_B\}\Big),$$

where

$$V = \{\, T_1,\, T_2,\, T_3\} \subseteq \mathcal{P}(V_0), \quad E = \{\, D_A,\, D_B\} \subseteq \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

In this example:

- *Each* level-0 *element (e.g., "Alice") is an employee.*
- *Each* level-1 *hyperedge (e.g., $T_1 = \{\textit{Alice}, \textit{Bob}\}$) is a project team consisting of two employees.*
- *Each* level-2 *supervertex (e.g., T_1) is viewed as a single "team-vertex" in the iterated powerset.*
- *Each* level-2 *superedge (e.g., $D_A = \{T_1, T_2\}$) is a department that oversees Team 1 and Team 2.*

Thus $\mathrm{SHG}^{(2)} = (V, E)$ accurately captures the hierarchical organization of employees → teams → departments as a 2-SuperHyperGraph.

1.2 Probabilistic graph and probabilistic hypergraph

Probability quantifies the chance of events occurring based on randomness or uncertainty in a system [22–27]. A *Probabilistic Graph* is a graph in which each edge carries an associated probability reflecting uncertainty in connections between vertices, thereby enabling probabilistic network analysis [28–31]. A *Probabilistic Hypergraph* is a hypergraph in which each hyperedge carries an associated probability reflecting uncertainty in multi-vertex relationships, thereby enabling probabilistic modeling of higher-order networks (cf.[32–35]).

Definition 1.9 (Probabilistic Graph). *(cf.[28–31]) A* Probabilistic Graph *is a triplet*

$$G = \big(V,\, E,\, A\big),$$

where:

- V *is a finite set of vertices.*
- $E \subseteq \big\{\{u,v\} \;\big|\; u,v \in V,\, u \neq v\big\}$ *is a set of (unordered) edges.*
- $A : V \times V \longrightarrow [0,1]$ *is an* affinity function *(or probability matrix) such that, for any* $u,v \in V$, $A(u,v)$ *represents the probability (or weight) of the edge* $\{u,v\}$ *existing.*

For each edge $e = \{u,v\} \in E$*, its weight is defined by*

$$w(e) \;=\; A(u,v).$$

The degree *of a vertex* $v \in V$ *is given by*

$$d(v) \;=\; \sum_{\substack{u \in V \\ \{v,u\} \in E}} A(v,u).$$

The adjacency matrix M *of* G *is the* $|V| \times |V|$ *matrix defined by*

$$M(i,j) \;=\; \begin{cases} A\big(v_i, v_j\big), & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.10 (Probabilistic Hypergraph). *(cf.[32–34]) A* Probabilistic Hypergraph *is a triplet*

$$H = \big(V,\, \mathcal{E},\, A\big),$$

where:

- V *is a finite set of vertices.*
- $\mathcal{E} \subseteq 2^V$ *is a collection of* hyperedges*, each hyperedge* $e \in \mathcal{E}$ *being a nonempty subset of* V*.*
- $A : V \times V \longrightarrow [0,1]$ *is an* affinity function *that assigns a probability (or similarity)* $A(u,v)$ *to each ordered pair of vertices* (u,v)*,* $u,v \in V$*.*

To each hyperedge $e \in \mathcal{E}$*, we associate a* centroid *vertex* $c(e) \in e$ *chosen according to a specified criterion (e.g., maximum total similarity):*

$$c(e) \;=\; \arg\max_{w \in e} \sum_{u \in e} A\big(w, u\big).$$

The incidence matrix H *of the probabilistic hypergraph is the* $|V| \times |\mathcal{E}|$ *matrix defined by*

$$H(v, e) \;=\; \begin{cases} A\big(c(e),\, v\big), & \text{if } v \in e, \\ 0, & \text{otherwise}. \end{cases}$$

The weight *of a hyperedge* $e \in \mathcal{E}$ *is*

$$w(e) \;=\; \sum_{v \in e} A\big(c(e),\, v\big).$$

The degree *of a vertex* $v \in V$ *is given by*

$$d(v) \;=\; \sum_{e \in \mathcal{E}} w(e)\, \big[H(v, e)\big],$$

and the degree *of a hyperedge* $e \in \mathcal{E}$ *is*

$$\delta(e) \;=\; \sum_{v \in e} H(v, e).$$

Example 1.11 (Real-World Example of a Probabilistic Hypergraph: Co-Authorship Network)**.** *Consider a research community in which each vertex represents an author, and each hyperedge represents the set of authors on a published paper. We model this as a* Probabilistic Hypergraph

$$H = \big(V,\, \mathcal{E},\, A\big),$$

as follows:

- ***Vertices.***

  $$V \;=\; \{\textit{Alice},\, \textit{Bob},\, \textit{Carol},\, \textit{Dave}\}.$$

  Each element of V *is a distinct researcher.*

- ***Hyperedges.*** *Suppose there are three recent multi-author papers:*

  $$e_1 \;=\; \{\textit{Alice},\, \textit{Bob},\, \textit{Carol}\}, \quad e_2 \;=\; \{\textit{Bob},\, \textit{Carol},\, \textit{Dave}\}, \quad e_3 \;=\; \{\textit{Alice},\, \textit{Dave}\}.$$

  Thus

  $$\mathcal{E} \;=\; \{\, e_1,\, e_2,\, e_3 \}, \quad e_1, e_2, e_3 \subseteq V.$$

- ***Affinity Function.*** *Define*

  $$A : V \times V \;\longrightarrow\; [0, 1],$$

  where $A(u, v)$ *is the estimated probability that authors* u *and* v *will collaborate on a future paper. For concreteness, suppose:*

  $$\begin{aligned} &A(\textit{Alice},\, \textit{Bob}) = 0.60, \quad A(\textit{Alice},\, \textit{Carol}) = 0.50, \quad A(\textit{Alice},\, \textit{Dave}) = 0.30, \\ &A(\textit{Bob},\, \textit{Carol}) = 0.70, \quad A(\textit{Bob},\, \textit{Dave}) = 0.40, \quad A(\textit{Carol},\, \textit{Dave}) = 0.45, \\ &A(x, x) = 0 \quad (\forall\, x \in V), \end{aligned}$$

  and symmetry $A(u, v) = A(v, u)$.

- ***Centroid of Each Hyperedge.*** *For each hyperedge $e \in \mathcal{E}$, choose the* centroid $c(e) \in e$ *as the author whose sum of affinities to the other authors in e is maximal:*

$$c(e) \;=\; \arg\max_{w \in e} \sum_{u \in e} A\big(w,\, u\big).$$

  - *For $e_1 = \{\text{Alice},\ \text{Bob},\ \text{Carol}\}$, compute*

$$\sum_{u \in e_1} A(\text{Alice}, u) = A(\text{Alice}, \text{Alice}) + A(\text{Alice}, \text{Bob}) + A(\text{Alice}, \text{Carol}) = 0 + 0.60 + 0.50 = 1.10,$$

$$\sum_{u \in e_1} A(\text{Bob}, u) = A(\text{Bob}, \text{Alice}) + A(\text{Bob}, \text{Bob}) + A(\text{Bob}, \text{Carol}) = 0.60 + 0 + 0.70 = 1.30,$$

$$\sum_{u \in e_1} A(\text{Carol}, u) = A(\text{Carol}, \text{Alice}) + A(\text{Carol}, \text{Bob}) + A(\text{Carol}, \text{Carol}) = 0.50 + 0.70 + 0 = 1.20.$$

    The maximum is 1.30 at Bob, so

$$c(e_1) = \text{Bob}.$$

  - *For $e_2 = \{\text{Bob},\ \text{Carol},\ \text{Dave}\}$, compute*

$$\sum_{u \in e_2} A(\text{Bob}, u) = A(\text{Bob}, \text{Bob}) + A(\text{Bob}, \text{Carol}) + A(\text{Bob}, \text{Dave}) = 0 + 0.70 + 0.40 = 1.10,$$

$$\sum_{u \in e_2} A(\text{Carol}, u) = A(\text{Carol}, \text{Bob}) + A(\text{Carol}, \text{Carol}) + A(\text{Carol}, \text{Dave}) = 0.70 + 0 + 0.45 = 1.15,$$

$$\sum_{u \in e_2} A(\text{Dave}, u) = A(\text{Dave}, \text{Bob}) + A(\text{Dave}, \text{Carol}) + A(\text{Dave}, \text{Dave}) = 0.40 + 0.45 + 0 = 0.85.$$

    The maximum is 1.15 at Carol, so

$$c(e_2) = \text{Carol}.$$

  - *For $e_3 = \{\text{Alice},\ \text{Dave}\}$, compute*

$$\sum_{u \in e_3} A(\text{Alice}, u) = A(\text{Alice}, \text{Alice}) + A(\text{Alice}, \text{Dave}) = 0 + 0.30 = 0.30,$$

$$\sum_{u \in e_3} A(\text{Dave}, u) = A(\text{Dave}, \text{Alice}) + A(\text{Dave}, \text{Dave}) = 0.30 + 0 = 0.30.$$

    They tie; choose

$$c(e_3) = \text{Alice}.$$

- ***Incidence Matrix.*** *Label the rows by the vertices $\{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}\}$ and the columns by the hyperedges $\{e_1, e_2, e_3\}$. Then*

$$H(v, e) = \begin{cases} A\big(c(e),\, v\big), & \text{if } v \in e, \\ 0, & \text{if } v \notin e. \end{cases}$$

  Concretely, Table 1 shows the values of H and A for each actor and event.

  Each entry $H(v, e)$ is zero if $v \notin e$.

Table 1
Example of H and A values for each actor and event

	e_1	e_2	e_3
Alice	$H(\text{Alice}, e_1) = A(\text{Bob}, \text{Alice}) = 0.60$	0	$H(\text{Alice}, e_3) = A(\text{Alice}, \text{Alice}) = 0$
Bob	$H(\text{Bob}, e_1) = A(\text{Bob}, \text{Bob}) = 0$	$H(\text{Bob}, e_2) = A(\text{Carol}, \text{Bob}) = 0.70$	0
Carol	$H(\text{Carol}, e_1) = A(\text{Bob}, \text{Carol}) = 0.70$	$H(\text{Carol}, e_2) = A(\text{Carol}, \text{Carol}) = 0$	0
Dave	0	$H(\text{Dave}, e_2) = A(\text{Carol}, \text{Dave}) = 0.45$	$H(\text{Dave}, e_3) = A(\text{Alice}, \text{Dave}) = 0.30$

- ***Hyperedge Weights.*** *For each hyperedge $e \in \mathcal{E}$,*

$$w(e) \;=\; \sum_{v \in e} A\big(c(e),\, v\big).$$

$$\begin{aligned} w(e_1) &= A\big(c(e_1) = \textit{Bob},\, \textit{Alice}\big) + A\big(\textit{Bob},\, \textit{Bob}\big) + A\big(\textit{Bob},\, \textit{Carol}\big) \\ &= 0.60 + 0 + 0.70 = 1.30, \end{aligned}$$

$$\begin{aligned} w(e_2) &= A\big(c(e_2) = \textit{Carol},\, \textit{Bob}\big) + A\big(\textit{Carol},\, \textit{Carol}\big) + A\big(\textit{Carol},\, \textit{Dave}\big) \\ &= 0.70 + 0 + 0.45 = 1.15, \end{aligned}$$

$$w(e_3) = A\big(c(e_3) = \textit{Alice},\, \textit{Alice}\big) + A\big(\textit{Alice},\, \textit{Dave}\big) = 0 + 0.30 = 0.30.$$

- ***Vertex Degrees.*** *For each vertex $v \in V$,*

$$d(v) \;=\; \sum_{e \in \mathcal{E}} w(e)\, \big[\, H(v, e)\big].$$

$$\begin{aligned} d(\textit{Alice}) &= w(e_1)\, H(\textit{Alice}, e_1) + w(e_2)\, H(\textit{Alice}, e_2) + w(e_3)\, H(\textit{Alice}, e_3) \\ &= 1.30 \cdot 0.60 + 1.15 \cdot 0 + 0.30 \cdot 0 = 0.78, \end{aligned}$$

$$d(\textit{Bob}) = 1.30 \cdot 0 + 1.15 \cdot 0.70 + 0.30 \cdot 0 = 0.805,$$

$$d(\textit{Carol}) = 1.30 \cdot 0.70 + 1.15 \cdot 0 + 0.30 \cdot 0 = 0.91,$$

$$d(\textit{Dave}) = 1.30 \cdot 0 + 1.15 \cdot 0.45 + 0.30 \cdot 0.30 = 0.5175 + 0.09 = 0.6075.$$

- ***Hyperedge Degrees.*** *For each hyperedge $e \in \mathcal{E}$,*

$$\delta(e) \;=\; \sum_{v \in e} H(v, e).$$

$$\delta(e_1) = H(\textit{Alice}, e_1) + H(\textit{Bob}, e_1) + H(\textit{Carol}, e_1) = 0.60 + 0 + 0.70 = 1.30,$$

$$\delta(e_2) = H(\textit{Bob}, e_2) + H(\textit{Carol}, e_2) + H(\textit{Dave}, e_2) = 0.70 + 0 + 0.45 = 1.15,$$

$$\delta(e_3) = H(\textit{Alice}, e_3) + H(\textit{Dave}, e_3) = 0 + 0.30 = 0.30.$$

In summary, the triplet

$$H \;=\; \Big(V,\, \mathcal{E},\, A\Big) \;=\; \Big(\{\textit{Alice},\, \textit{Bob},\, \textit{Carol},\, \textit{Dave}\},\, \{e_1, e_2, e_3\},\, A\Big),$$

together with its incidence matrix $H(v, e)$, hyperedge weights $w(e)$, and degrees $d(v)$ and $\delta(e)$, constitutes a concrete Probabilistic Hypergraph *modeling collaboration probabilities in a small co-authorship network.*

2. Result: Probabilistic n-SuperHyperGraph

A Probabilistic n-SuperHyperGraph extends probabilistic hypergraphs to nested powerset levels, assigning probabilities to n-level superedges for complex hierarchical uncertainty modeling(cf.[36]).

Definition 2.1 (Probabilistic n-SuperHyperGraph)**.** *(cf.[36]) Let V_0 be a finite base set, and let $\mathcal{P}^n(V_0)$ be the n-th iterated powerset as in Definition 1.7.* A Probabilistic n-SuperHyperGraph *is a triplet*

$$G = (V, E, A),$$

where:

- $V \subseteq \mathcal{P}^n(V_0)$ *is a finite set of* n-supervertices.
- $E \subseteq \mathcal{P}^n(V_0)$ *is a finite set of* n-superedges, *each* $e \in E$ *being a subset of* $\mathcal{P}^n(V_0)$.
- $A : V \times V \longrightarrow [0, 1]$ *is an* affinity function *(probability function) on pairs of n-supervertices.*

To each n-superedge $e \in E$, select a centroid supervertex $c(e) \in e$ *according to a chosen criterion (for example, similarity-based):*

$$c(e) = \arg\max_{w \in e} \sum_{u \in e} A(w, u).$$

The incidence matrix H *of G is the $|V| \times |E|$ matrix defined by*

$$H(v, e) = \begin{cases} A(c(e), v), & \text{if } v \in e, \\ 0, & \text{otherwise}. \end{cases}$$

For each superedge $e \in E$, its weight *is*

$$w(e) = \sum_{v \in e} A(c(e), v).$$

The degree *of a supervertex $v \in V$ is*

$$d(v) = \sum_{e \in E} w(e) \, [H(v, e)],$$

and the degree *of a superedge $e \in E$ is*

$$\delta(e) = \sum_{v \in e} H(v, e).$$

Example 2.2 (Real-World Example of a Probabilistic 2-SuperHyperGraph: Smart Building Sensor Network)**.** *Consider a smart building equipped with multiple wireless sensors. We model the hierarchy as follows:*

- *The base set of* sensors *(level-0 vertices) is*

$$V_0 = \{s_1, s_2, s_3, s_4, s_5, s_6\},$$

where each s_i is a temperature or motion sensor in the building.

- *First, form the* hyperedges *(level-*1*) by grouping sensors into* subnetworks*:*

$$X_1 = \{s_1, s_2\}, \quad X_2 = \{s_3, s_4\}, \quad X_3 = \{s_5, s_6\}.$$

Each $X_j \subseteq V_0$ *is a subset of sensors that share a local gateway.*

- *Next, form the* level-2 supervertices *by taking subsets of* $\{X_1, X_2, X_3\}$*. We select:*

$$V = \{X_1, X_2, X_3\} \subseteq \mathcal{P}(V_0) \quad \subseteq \quad \mathcal{P}(\mathcal{P}(V_0)).$$

- *Finally, choose two* level-2 superedges *(each an element of* $\mathcal{P}(\{X_1, X_2, X_3\})$*):*

$$e_1 = \{X_1, X_2\}, \qquad e_2 = \{X_2, X_3\}.$$

We now define the affinity function A *on the set of level-*2 *supervertices* $V = \{X_1, X_2, X_3\}$*, where* $A(X_i, X_j)$ *represents the probability that the corresponding subnetworks communicate reliably:*

$$A(X_1, X_2) = 0.80, \quad A(X_2, X_3) = 0.60, \quad A(X_1, X_3) = 0.40, \quad A(X_i, X_i) = 0 \quad (\forall\, i).$$

Thus, (V, E, A) *satisfies the data of a* Probabilistic 2-SuperHyperGraph *as in Definition 2.1.*

Centroid selection for each superedge. *For each* $e_k \in \{e_1, e_2\}$*, choose the* centroid supervertex $c(e_k) \in e_k$ *by maximizing the sum of affinities within that superedge.*

- *For* $e_1 = \{X_1, X_2\}$*:*

$$\sum_{u \in e_1} A(X_1, u) = A(X_1, X_1) + A(X_1, X_2) = 0 + 0.80 = 0.80,$$

$$\sum_{u \in e_1} A(X_2, u) = A(X_2, X_1) + A(X_2, X_2) = 0.80 + 0 = 0.80.$$

Both sums coincide; by convention, we set

$$c(e_1) = X_1.$$

- *For* $e_2 = \{X_2, X_3\}$*:*

$$\sum_{u \in e_2} A(X_2, u) = A(X_2, X_2) + A(X_2, X_3) = 0 + 0.60 = 0.60,$$

$$\sum_{u \in e_2} A(X_3, u) = A(X_3, X_2) + A(X_3, X_3) = 0.60 + 0 = 0.60.$$

Again a tie; we choose

$$c(e_2) = X_2.$$

Incidence matrix. *Label the supervertices in* V *as* X_1, X_2, X_3 *(rows) and the superedges as* e_1, e_2 *(columns). Then the incidence entry* $H(X_i, e_k)$ *is*

$$H(X_i, e_k) = \begin{cases} A(c(e_k), X_i), & \text{if } X_i \in e_k, \\ 0, & \text{otherwise}. \end{cases}$$

Table 2
Entries: $H(X_i, e_k)$

	e_1	e_2
X_1	$A\big(c(e_1) = X_1, X_1\big) = 0$	0
X_2	$A\big(c(e_1) = X_1, X_2\big) = 0.80$	$A\big(c(e_2) = X_2, X_2\big) = 0$
X_3	0	$A\big(c(e_2) = X_2, X_3\big) = 0.60$

Hence, Table 2 presents the values of $H(X_i, e_k)$ for each combination of X_i and e_k.

Hyperedge weights. *For each e_k, its weight is*

$$w(e_k) \;=\; \sum_{X_i \in e_k} A\big(c(e_k),\, X_i\big).$$

$$w(e_1) = A\big(c(e_1) = X_1,\, X_1\big) \;+\; A\big(c(e_1) = X_1,\, X_2\big) = 0 + 0.80 = 0.80,$$
$$w(e_2) = A\big(c(e_2) = X_2,\, X_2\big) \;+\; A\big(c(e_2) = X_2,\, X_3\big) = 0 + 0.60 = 0.60.$$

Vertex (supervertex) degrees. *For each $X_i \in V$,*

$$d(X_i) \;=\; \sum_{e \in \{e_1, e_2\}} w(e)\, \big[\, H(X_i,\, e)\big].$$

$$d(X_1) = w(e_1)\, H(X_1, e_1) + w(e_2)\, H(X_1, e_2) = 0.80 \cdot 0 + 0.60 \cdot 0 = 0,$$
$$d(X_2) = w(e_1)\, H(X_2, e_1) + w(e_2)\, H(X_2, e_2) = 0.80 \cdot 0.80 + 0.60 \cdot 0 = 0.64,$$
$$d(X_3) = w(e_1)\, H(X_3, e_1) + w(e_2)\, H(X_3, e_2) = 0.80 \cdot 0 + 0.60 \cdot 0.60 = 0.36.$$

Superedge degrees. *For each e_k,*

$$\delta(e_k) \;=\; \sum_{X_i \in e_k} H(X_i,\, e_k).$$

$$\delta(e_1) \;=\; H(X_1, e_1) + H(X_2, e_1) \;=\; 0 + 0.80 = 0.80,$$
$$\delta(e_2) \;=\; H(X_2, e_2) + H(X_3, e_2) \;=\; 0 + 0.60 = 0.60.$$

In summary, the triplet

$$\big(V,\, E,\, A\big) \;=\; \big(\{X_1, X_2, X_3\},\, \{e_1, e_2\},\, A\big)$$

with the above incidence matrix H, weights $w(e_k)$, and degrees $d(X_i), \delta(e_k)$ constitutes a concrete Probabilistic 2-SuperHyperGraph *modeling sensor-group communication reliability in a smart building*.

Example 2.3 (Real-World Example of a Probabilistic 2-SuperHyperGraph: Gene–Pathway–Superpathway Network)**.** *Consider a biological network where individual genes form pathways, and pathways in turn form higher-order "superpathways." We model this as a Probabilistic 2-SuperHyperGraph by taking:*

- *The base set of* genes *(level-0 vertices):*

$$V_0 = \{\, g_1,\, g_2,\, g_3,\, g_4,\, g_5,\, g_6,\, g_7,\, g_8\},$$

  where each g_i is a protein-coding gene.

- *Level-1* pathways *(hyperedges of genes) as subsets of V_0:*

$$P_1 = \{\, g_1,\, g_2,\, g_3\}, \quad P_2 = \{\, g_4,\, g_5,\, g_6\}, \quad P_3 = \{\, g_6,\, g_7,\, g_8\}.$$

  Each $P_j \subseteq V_0$ is a known biological pathway (e.g., metabolic or signaling).

- *Level-2* superpathways *(supervertices) as subsets of $\{P_1, P_2, P_3\}$:*

$$V = \{\, P_1,\, P_2,\, P_3\} \;\subseteq\; \mathcal{P}(V_0) \;\subseteq\; \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

- *Level-2* superedges *linking collections of pathways that co-regulate higher-order functions:*

$$e_1 = \{\, P_1,\, P_2\}, \qquad e_2 = \{\, P_2,\, P_3\}.$$

Next, define an affinity function

$$A : V \times V \;\longrightarrow\; [0,1],$$

where $A(P_i, P_j)$ represents the estimated probability that pathways P_i and P_j co-occur or co-regulate in the same cellular condition (e.g., based on co-expression data). Suppose:

$$A(P_1, P_2) = 0.75, \quad A(P_2, P_3) = 0.55, \quad A(P_1, P_3) = 0.20, \quad A(P_i, P_i) = 0,\ \forall i.$$

Thus $(V, \{e_1, e_2\}, A)$ forms a Probabilistic 2-SuperHyperGraph as in Definition 2.1.

Centroid selection for each superedge. *For each superedge e_k, choose a* centroid pathway $c(e_k)$ *by maximizing the sum of affinities within that superedge:*

$$c(e_k) \;=\; \arg\max_{P \in e_k} \sum_{Q \in e_k} A(P, Q).$$

- For $e_1 = \{\, P_1, P_2\}$:

$$\sum_{Q \in e_1} A(P_1, Q) = A(P_1, P_1) + A(P_1, P_2) = 0 + 0.75 = 0.75,$$

$$\sum_{Q \in e_1} A(P_2, Q) = A(P_2, P_1) + A(P_2, P_2) = 0.75 + 0 = 0.75.$$

  Tied sums; choose

$$c(e_1) = P_1.$$

- For $e_2 = \{\, P_2, P_3\}$:

$$\sum_{Q \in e_2} A(P_2, Q) = A(P_2, P_2) + A(P_2, P_3) = 0 + 0.55 = 0.55,$$

$$\sum_{Q \in e_2} A(P_3, Q) = A(P_3, P_2) + A(P_3, P_3) = 0.55 + 0 = 0.55.$$

  Tied again; choose

$$c(e_2) = P_2.$$

Incidence matrix. *Index the pathways* P_1, P_2, P_3 *as rows and the superedges* e_1, e_2 *as columns. Then*

$$H\big(P_i,\, e_k\big) = \begin{cases} A\big(c(e_k),\, P_i\big), & \text{if } P_i \in e_k, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, Table 3 shows the values of $H(P_i, e_k)$ *for each provider and event.*

Table 3
Values of $H(P_i, e_k)$ for each provider and event

	e_1	e_2
P_1	$A\big(c(e_1) = P_1, P_1\big) = 0$	0
P_2	$A\big(c(e_1) = P_1, P_2\big) = 0.75$	$A\big(c(e_2) = P_2, P_2\big) = 0$
P_3	0	$A\big(c(e_2) = P_2, P_3\big) = 0.55$

Superedge weights. *For each* $e_k \in \{e_1, e_2\}$,

$$w(e_k) \;=\; \sum_{P \in e_k} A\big(c(e_k),\, P\big).$$

$$w(e_1) = A\big(c(e_1) = P_1,\, P_1\big) \;+\; A\big(c(e_1) = P_1,\, P_2\big) = 0 + 0.75 = 0.75,$$
$$w(e_2) = A\big(c(e_2) = P_2,\, P_2\big) \;+\; A\big(c(e_2) = P_2,\, P_3\big) = 0 + 0.55 = 0.55.$$

Pathway degrees. *For each* $P_i \in \{P_1, P_2, P_3\}$,

$$d(P_i) \;=\; \sum_{e \in \{e_1, e_2\}} w(e)\, \big[\, H(P_i,\, e)\big].$$

$$d(P_1) = w(e_1)\, H(P_1, e_1) + w(e_2)\, H(P_1, e_2) = 0.75 \cdot 0 + 0.55 \cdot 0 = 0,$$
$$d(P_2) = w(e_1)\, H(P_2, e_1) + w(e_2)\, H(P_2, e_2) = 0.75 \cdot 0.75 + 0.55 \cdot 0 = 0.5625,$$
$$d(P_3) = w(e_1)\, H(P_3, e_1) + w(e_2)\, H(P_3, e_2) = 0.75 \cdot 0 + 0.55 \cdot 0.55 = 0.3025.$$

Superedge degrees. *For each superedge* e_k,

$$\delta(e_k) \;=\; \sum_{P \in e_k} H(P,\, e_k).$$

$$\delta(e_1) = H(P_1, e_1) + H(P_2, e_1) = 0 + 0.75 = 0.75, \quad \delta(e_2) = H(P_2, e_2) + H(P_3, e_2) = 0 + 0.55 = 0.55.$$

In summary, the triplet

$$\big(V,\, E,\, A\big) = \big(\{P_1, P_2, P_3\},\, \{e_1, e_2\},\, A\big),$$

equipped with the incidence matrix H, *weights* $w(e_k)$, *and degrees* $d(P_i)$, $\delta(e_k)$, *constitutes a concrete* Probabilistic 2-SuperHyperGraph *modeling co-regulation among gene pathways.*

Theorem 2.4 (Structural Preservation). *Every Probabilistic n-SuperHyperGraph $G = (V, E, A)$ is an n-SuperHyperGraph when forgetting the affinity function A. In particular, the pair*

$$H = (V,\ E)$$

satisfies Definition 1.7.

Proof. By assumption, V and E are both subsets of $\mathcal{P}^n(V_0)$, where V_0 is the finite base set of Definition 1.7. Hence, simply discarding the affinity function $A : V \times V \to [0, 1]$ leaves the underlying pair (V, E), which by construction satisfies

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0).$$

Therefore, (V, E) is exactly an n-SuperHyperGraph in the sense of Definition 1.7. This shows that the probabilistic enhancement does not alter the pure combinatorial structure of an n-SuperHyperGraph. □

Theorem 2.5 (Reduction to Probabilistic Hypergraph). *When $n = 0$, a Probabilistic n-SuperHyperGraph $G = (V, E, A)$ reduces to an ordinary Probabilistic Hypergraph. Specifically, if $n = 0$ then $V \subseteq V_0$, $E \subseteq \mathcal{P}(V_0)$, and G coincides with Definition 1.10.*

Proof. By Definition 1.7, if $n = 0$ then $\mathcal{P}^0(V_0) = V_0$. Consequently,

$$V \subseteq \mathcal{P}^0(V_0) = V_0, \quad E \subseteq \mathcal{P}^0(V_0) = V_0.$$

Since each $e \in E$ is a subset of $\mathcal{P}^0(V_0) = V_0$, we have $E \subseteq \mathcal{P}(V_0)$. The affinity function

$$A : V \times V \ \longrightarrow\ [0, 1]$$

then plays exactly the role of the similarity/probability matrix in Definition 1.10. The incidence matrix H defined by

$$H(v, e) = \begin{cases} A\big(c(e),\ v\big), & \text{if } v \in e, \\ 0, & \text{otherwise}, \end{cases}$$

and the corresponding definitions of $w(e)$, $d(v)$, and $\delta(e)$ coincide with those of a Probabilistic Hypergraph (Definition 1.10). Hence G is exactly a Probabilistic Hypergraph in this case. □

Theorem 2.6 (Closure). *Let $G = (V, E, A)$ be a Probabilistic n-SuperHyperGraph. For any subset $V' \subseteq V$, define*

$$E' \ = \ \{ e \in E \ \big|\ e \subseteq V' \}, \quad A' \ = \ A\big|_{V' \times V'}.$$

Then

$$G' \ = \ \big(V',\ E',\ A'\big)$$

is also a Probabilistic n-SuperHyperGraph.

Proof. Since $V' \subseteq V \subseteq \mathcal{P}^n(V_0)$, we have $V' \subseteq \mathcal{P}^n(V_0)$. Moreover,

$$E' \ = \ \{ e \in E \mid e \subseteq V' \} \ \subseteq \ E \ \subseteq \ \mathcal{P}^n(V_0).$$

Thus $V' \subseteq \mathcal{P}^n(V_0)$ and $E' \subseteq \mathcal{P}^n(V_0)$, so (V', E') satisfies the requirements of an n-SuperHyperGraph (Definition 1.7). The restriction $A' = A|_{V' \times V'}$ remains a valid affinity function

$$A' : V' \times V' \ \longrightarrow\ [0, 1],$$

and one defines the incidence matrix H' for G' by

$$H'(v,e) = \begin{cases} A'\big(c(e),\, v\big), & \text{if } v \in e,\ e \in E', \\ 0, & \text{otherwise}. \end{cases}$$

Since each $e \in E'$ satisfies $e \subseteq V'$ and $c(e) \in e \subseteq V'$, the centroid $c(e)$ is well–defined in V'. The definitions of superedge weight $w'(e)$, supervertex degree $d'(v)$, and superedge degree $\delta'(e)$ mirror those in G. Hence $G' = (V', E', A')$ is again a Probabilistic n-SuperHyperGraph. □

Theorem 2.7 (Degree–Sum Identity). *In any Probabilistic n-SuperHyperGraph $G = (V, E, A)$, the sum of all supervertex degrees equals the sum of superedge weights times their superedge degrees:*

$$\sum_{v\in V} d(v) \;=\; \sum_{e\in E} w(e)\,\delta(e).$$

Proof. By definition, for each $v \in V$,

$$d(v) \;=\; \sum_{e\in E} w(e)\,H(v,e).$$

Summing over all $v \in V$ gives

$$\sum_{v\in V} d(v) \;=\; \sum_{v\in V}\sum_{e\in E} w(e)\,H(v,e).$$

Since $w(e)$ does not depend on v, we interchange the sums:

$$\sum_{v\in V}\sum_{e\in E} w(e)\,H(v,e) \;=\; \sum_{e\in E} w(e)\,\sum_{v\in V} H(v,e).$$

By Definition 2.1, for each $e \in E$,

$$\sum_{v\in V} H(v,e) \;=\; \sum_{v\in e} H(v,e) \;=\; \delta(e).$$

Therefore,

$$\sum_{v\in V} d(v) \;=\; \sum_{e\in E} w(e)\,\delta(e),$$

as claimed. □

Theorem 2.8 (Upper Bound on Superedge Weight). *Let $e \in E$ be any n-superedge. Then*

$$w(e) \;=\; \sum_{v\in e} A\big(c(e),\, v\big) \;\le\; |e|,$$

where $|e|$ denotes the cardinality of the set e. In particular, if $A(c(e), v) \le 1$ for all $v \in e$, then $w(e) \le |e|$.

Proof. By hypothesis, $A : V \times V \to [0, 1]$. Hence for every $v \in e$,

$$0 \;\le\; A\big(c(e),\, v\big) \;\le\; 1.$$

Summing this inequality over all $v \in e$ yields

$$0 \;\le\; \sum_{v\in e} A\big(c(e),\, v\big) \;\le\; \sum_{v\in e} 1 \;=\; |e|.$$

Since the left–hand sum is precisely $w(e)$ by definition, we obtain $w(e) \le |e|$. □

Theorem 2.9 (Monotonicity under Fixed Centroid). *Let $e_1, e_2 \in E$ be two n-superedges satisfying*

$$e_1 \subseteq e_2 \quad \textit{and} \quad c(e_1) \;=\; c(e_2) =: c.$$

Then

$$w(e_1) \;\leq\; w(e_2).$$

Proof. By assumption, $c(e_1) = c(e_2) = c$. Then

$$w(e_1) \;=\; \sum_{v \in e_1} A\big(c,\, v\big), \quad w(e_2) \;=\; \sum_{v \in e_2} A\big(c,\, v\big).$$

Since $e_1 \subseteq e_2$, each term $A(c, v)$ with $v \in e_1$ also appears in the sum for $w(e_2)$. Moreover, all terms $A(c, v)$ are nonnegative. Therefore,

$$\sum_{v \in e_1} A\big(c,\, v\big) \;\leq\; \sum_{v \in e_2} A\big(c,\, v\big),$$

which is precisely $w(e_1) \leq w(e_2)$. □

Theorem 2.10 (Projection to Level-$(n-1)$). *Suppose $G = (V, E, A)$ is a Probabilistic n-SuperHyperGraph with*

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0).$$

Define the level-$(n-1)$ projection *of a supervertex $v \in V$ by*

$$\pi_{n-1}(v) \;=\; \big\{\, u \in \mathcal{P}^{n-1}(V_0) \;\big|\; u \in v \big\},$$

and similarly, for a superedge $e = \{v_1, \ldots, v_k\} \in E$, set

$$\pi_{n-1}(e) \;=\; \big\{\pi_{n-1}(v_1),\, \ldots,\, \pi_{n-1}(v_k)\big\}.$$

Let

$$V' \;=\; \big\{\pi_{n-1}(v) \;\big|\; v \in V\big\} \subseteq \mathcal{P}^{n-1}(V_0), \quad E' \;=\; \big\{\pi_{n-1}(e) \;\big|\; e \in E\big\} \subseteq \mathcal{P}^{n-1}(V_0).$$

Then there exists an induced affinity function

$$A' : V' \times V' \;\longrightarrow\; [0,1], \quad A'\big(\pi_{n-1}(u),\, \pi_{n-1}(v)\big) \;=\; A(u, v),$$

making

$$G' \;=\; \big(V',\, E',\, A'\big)$$

a Probabilistic $(n-1)$-SuperHyperGraph.

Proof. Since each $v \in V$ lies in $\mathcal{P}^n(V_0)$, it is a subset of $\mathcal{P}^{\,n-1}(V_0)$, so

$$\pi_{n-1}(v) \;=\; v \cap \mathcal{P}^{n-1}(V_0) \;\subseteq\; \mathcal{P}^{n-1}(V_0).$$

Thus $V' \subseteq \mathcal{P}^{n-1}(V_0)$. Similarly, if $e = \{v_1, \ldots, v_k\} \in E$, then each $v_i \in V$ yields $\pi_{n-1}(v_i) \subseteq \mathcal{P}^{n-1}(V_0)$, so

$$\pi_{n-1}(e) \;=\; \{\pi_{n-1}(v_1), \ldots, \pi_{n-1}(v_k)\} \;\subseteq\; \mathcal{P}^{n-1}(V_0).$$

Hence $E' \subseteq \mathcal{P}^{n-1}(V_0)$. By defining

$$A'\big(\pi_{n-1}(u),\, \pi_{n-1}(v)\big) = A(u, v), \quad \forall\, u, v \in V,$$

we obtain a well-defined affinity $A' : V' \times V' \to [0,1]$, since π_{n-1} is surjective onto V'. For each projected superedge $\pi_{n-1}(e) \in E'$, choose its centroid

$$c'\big(\pi_{n-1}(e)\big) \;=\; \pi_{n-1}\big(c(e)\big),$$

noting that $c(e) \in e$ implies $\pi_{n-1}(c(e)) \in \pi_{n-1}(e)$. Then define the incidence matrix $H' : V' \times E' \to [0,1]$ by

$$H'\big(u,\, \pi_{n-1}(e)\big) = \begin{cases} A'\big(c'(\pi_{n-1}(e)),\, u\big), & \text{if } u \in \pi_{n-1}(e), \\ 0, & \text{otherwise}. \end{cases}$$

This construction satisfies all requirements of a Probabilistic $(n-1)$-SuperHyperGraph, as $V' \subseteq \mathcal{P}^{n-1}(V_0)$, $E' \subseteq \mathcal{P}^{n-1}(V_0)$, and A' is an affinity on V'. Therefore, $G' = (V', E', A')$ is a Probabilistic $(n-1)$-SuperHyperGraph. □

Theorem 2.11 (Symmetry of Incidence Sum). *Suppose that the affinity function $A : V \times V \to [0,1]$ is symmetric, i.e. $A(u,v) = A(v,u)$ for all $u, v \in V$. Then for every superedge $e \in E$ with centroid $c(e)$,*

$$w(e) \;=\; \sum_{v \in e} A\big(c(e),\, v\big) \;=\; \sum_{v \in e} A\big(v,\, c(e)\big).$$

Consequently,

$$w(e) \;=\; \delta(e)\, \overline{A}(e),$$

where

$$\overline{A}(e) \;=\; \frac{1}{\delta(e)} \sum_{v \in e} A\big(v,\, c(e)\big)$$

is the average affinity between the centroid and the other vertices in e.

Proof. Since A is symmetric, $A\big(c(e), v\big) = A(v, c(e))$ for all $v \in V$. By definition,

$$w(e) \;=\; \sum_{v \in e} A\big(c(e),\, v\big) \;=\; \sum_{v \in e} A\big(v,\, c(e)\big).$$

On the other hand,

$$\delta(e) \;=\; \sum_{v \in e} H(v, e) \;=\; \sum_{v \in e} A\big(c(e),\, v\big) \;=\; w(e).$$

Hence

$$\overline{A}(e) = \frac{1}{\delta(e)} \sum_{v \in e} A(v, c(e)),$$

and multiplying both sides by $\delta(e)$ gives

$$w(e) \;=\; \delta(e)\, \overline{A}(e).$$

□

3. Conclusion and future works

In this paper, we presented a concise investigation of the mathematical properties of Probabilistic n-SuperHyperGraphs and discussed their potential to advance the field of probabilistic network modeling. As future work, we intend to extend this framework by incorporating additional uncertainty-handling paradigms such as Fuzzy Sets [37, 38], Rough Sets[39], Intuitionistic Fuzzy Sets[40], Bipolar Fuzzy Sets[41, 42], HyperFuzzy Sets[43, 44], Hesitant Fuzzy Sets[45], Picture Fuzzy Sets[46], Neutrosophic Sets [47, 48], and Plithogenic Sets [49, 50]. We also aim to explore further generalizations using concepts such as HyperProbability and SuperHyperProbability [22, 24], which may provide a more comprehensive framework for modeling complex probabilistic structures.

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Conflicts of Interest

The authors declare no conflicts of interest.

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