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Decision-Making Framework for Urban Transportation Using Linear Diophantine Fuzzy Z-numbers with Dombi Aggregation, TOPSIS and VIKOR methods

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ABSTRACT

This study proposes an innovative approach to decision-making under uncertainty, using Pakistan's urban transportation issues—particularly in cities like Karachi, Lahore, and Islamabad—as a case study. These cities face severe traffic congestion, demanding more effective strategies for infrastructure planning. We introduce a Linear Diophantine Fuzzy Z-Numbers (LDFZN) framework that captures membership and non-membership grades alongside the degree of reliability, addressing key limitations of traditional fuzzy systems by simultaneously considering uncertainty and confidence. Within this framework, we develop three decision-making methods: an LDFZN Dombi Weighted Averaging operator that aggregates expert opinions while accounting for their reliability; an adapted VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method for multi-criteria compromise solutions under LDFZN settings; and a modified Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) tailored for LDFZN-based scenarios. These tools are applied to real-world transportation challenges in Pakistan, demonstrating their effectiveness in managing uncertainty and expert-based confidence levels. The results outperform conventional models in decision robustness and clarity under uncertain conditions. This work contributes significantly to the theoretical and practical advancement of fuzzy mathematics, extending uncertainty modeling and providing practical solutions not only for transportation but also for various fields requiring informed decision-making with imprecise and unreliable data.

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1. Introduction

An expansion of traditional real numbers, fuzzy numbers express quantities with imperfect values instead of precise integers. In fuzzy logic and fuzzy systems, they are widely utilized and crucial for modeling uncertainty. Numerous shapes, mainly triangular and trapezoidal, can be used to classify fuzzy numbers, which make arithmetic operations and applications in fuzzy analysis and optimization easier. Further details on their definitions, characteristics, and uses are provided in the sections that follow. Each value in a range is given a degree of truth by a membership function, which makes fuzzy numbers fuzzy subsets of the real line[1]. They enable the depiction of uncertainty in numerical values by generalizing closed intervals and real numbers[2]. The extension principle and a-cut with interval arithmetic are two techniques used in the arithmetic of fuzzy numbers [3]. The membership function shape of the T1SM affects the addition, multiplication, and division, which can lead to computations being more complex when they are not regular [4]. Fuzzy numbers are important in fuzzy optimization and differential equations and they provide a framework for epistemic uncertainty management in several domains [5]. They enhance the stability of systems in an uncertain environment and are employed in decision-making applications in the absence of precise information[6]. However, fuzzy numbers as a means to handle uncertainty are complex to calculate and interpret, particularly in cases of non-uniform membership functions. In some cases, when a simpler model would be desirable to use, the complexity of other models can become a barrier against their application. By combining fuzzy logic and linear Diophantine equations, linear Diophantine fuzzy numbers (LDFNs) are an extension of conventional fuzzy numbers that enable the representation of uncertainty in decision-making. In multi-objective optimization situations, these figures are especially helpful since they facilitate the efficient modeling of uncertain parameters. The qualities, uses, and importance of these are explained in further detail in the sections that follow. To represent the uncertain information by using parameters, the LDFNs fuse fuzzy sets and linear Diophantine equations.

LDFNs are stratified into three categories, with individual properties and score functions, including non-linear and improved scoring functions. In[7], LDFNs are applied to generate the solution for non-linear programming problems by facilitating the optimization of competing objectives under uncertainty. In the field of graph theory, shortest path problems are associated with weights in directed networks and better algorithms, such as Dijkstra's can be applied for better network routing decisions [8]. Emergency Decision Making: Making use of the newly defined similarity measures, LDFNs provide credible support for making decisions in emergency and incomplete information systems, to reduce uncertainty. Since LDFNs were introduced, a number of similarity and distance measures have been developed. These measures are essential for applications in domains such as data mining and medical diagnosis[9]. According to [7], LDFNs illustrate its adaptability in optimization contexts by providing a basis for addressing intricate nonlinear fractional programming problems. Although LDFNs have a lot to offer in terms of modeling uncertainty, their intricacy may make them difficult to use in real-world situations, so more study is needed to make their implementation easier. Dombi operations are a class of aggregation operators that, by offering flexibility through operational parameters, improve decision-making processes in fuzzy settings. Many fuzzy frameworks, such as m-polar image fuzzy, Pythagorean fuzzy, and complex Pythagorean fuzzy sets, have incorporated these operations, which include t-norms and t-conorms. There are three different kinds of operators: hybrid weighted average, order weighted average, and weighted average. Provided as an effective solution to solve challenging decision problems with the application in multiple attribute decision-making (MADM)[10]. An operator variant is such as geometric operators and mPoPF Dombi weighted averaging. It proved useful in MADM with the examples of using it as a means of selecting optimum locations of gas stations [11]. Additional operations in terms of aggregation are presented to regulate symmetry of decision making. Applied into the situation of selecting an expert to use proficiently, it proved its excellence compared to the traditional methods [12]. Such aspects are covered as operator development: complex weighted arithmetic and geometric averaging operators. The given example of the case of decision-making sheds light on the practical effect of bank decisions [13]. The problem of synthesizing data derived across different sources and expressing it through one singular value representative of all the information is one of the challenges in many fields, including computer science and engineering, economics, and the social sciences. When the data is complex as is usually the case and is contradicting to a large extent, it is important to ensure systematic means of efficiently combing information in order to analyze, interpret and make decisions. An important mathematical concept underthis procedure is the aggregation operator. An aggregation operator is a mapping (or simply a function) that takes a tuple of argument values to a single output value within the same domain, possibly a subset of the real numbers over a given interval, say between 0 and 1. This operation is supposed to represent or sum up the collection of the inputs. According to predetermined criteria, these operators are the fundamental mechanism for combining data points, expert opinions, sensor measurements, or preference ratings. The wide range of applications for aggregation operators highlights their significance and pervasiveness. According to Yager (1988), they are fundamental to multi-criteria decision making (MCDM), in which options are assessed according to several criteria and the performance scores are combined by an aggregation function to get an overall utility value[14]. The information fusion mostly depends on aggregation to combine input from multiple sensors or sources into a logical whole[15]. To combine fuzzy set memberships and implement logical connectives, aggregation operators—such as t-norms and t-conorms—are crucial in fuzzy logic and systems [16].They can also be used for database querying, machine learning for ensemble approaches, image processing, and pattern identification. It is frequently necessary to assess several, frequently opposing, criteria at once when navigating complex issues[17]. A methodical and structured technique to solving such issues is offered by (MCDM)[18]. MCDM techniques are a means of building given structures to assess a group of options with the notion of producing additional sensible and rational judgments as opposed to trusting in lone intuition. TOPSIS (Technique of Order of Preference by Similarity to Ideal Solution) is among the most used MCDM techniques and may be distinguished by simple logics and efficiency of the calculations[19]. TOPSIS, which was the brainchild of Hwang and Yoon, makes up two artificial reference points, including the negative-ideal solution, marking the worst performance, and the ideal solution, which marks the utmost performance on all criteria[20]. The basic assumption based on Behzadian, is that the best alternative must have maximum geometric distance between negative-ideal solution and minimum geometric distance between the ideal solution[21]. TOPSIS provides a complete ranking by calculating a relative proximity coefficient of each of the alternatives using these distances which helps a decision-maker to select an optimal alternative of choice[22]. The analytical approaches needed to address decision-making are more effective in the modern world. We apply and enhance the MCDM methods, particularly VIKOR and TOPSIS, towards addressing complex assessment scenarios that bear conflicting factors, in this research. We have found that the existing approaches to fuzzy do not suit well to the twofold purpose of measuring the uncertainty and demonstrating the reliability, which is a break in the critical decision making contexts. Our adventurous approach is based on how we can develop LDFZNs, Linear Diophantine Fuzzy Z-Numbers. The reliability of Z-numbers and the integrity of the theoretical underpinning of fuzzy sets that are linear, Diophantine are used by us. This method can be used to gather details of uncertainty and religious nature of decision. We have devised and provided two new aggregation operators, (1) the LDFZN Weighted Averaging (LDFZNWA) operator on main aggregation operators and (2) the LDFZN Dombi Weighted Averaging (LDFZNDWA) operator, with adaptive Dombi operators.In our work we achieved 3 key aims. The first thing was to develop full formal mathematical definitions of our suggested operators with actual proofs of their functions. Second, we presented detailed comparative study that demonstrated the superiority of our approaches regarding the ability to handle complicated situation of uncertainty in data compared to classic VIKOR and TOPSIS formulations. Third, we used these innovations in different areas, and they demonstrated their usefulness in supply chain resilience and AI-based risk management. With our contributions, we improve theoretical and practical decision science. We equip the decision-makers with more reliable means of operating in the turbulent environments by employing enhanced techniques. The current study proofs that we managed to incorporate the latest fuzzy theory with practice MCDM requirements, which resulted into the quantifiable increase in reliability and accuracy of decision-making.

2. Structure of article

The content and structure of the research article are organized in a way that there is proper flow towards the theoretical and practical application. It starts by introduction and abstract which describe the context and inspiration behind the study. These sections are followed by the preliminaries where a dedication to all the needed definitions and notations of fuzzy sets, Z-numbers, and Linear Diophantine constructs is provided in details. All this background sets the reader on a vibrant platform to await the main results section in which the proposed Linear Diophantine Fuzzy Z-Numbers Dombi Weighted Averaging (LDFZNDWA) operator will be brought in together with its mathematical characteristics and accompanying theorems. The article next proceeds to the statement of the problem, the specification of a real-world setting of decision making in the framework of the issues of transport in the urban context of Pakistan. Then, three stand-alone decision-making algorithms of an operator-based model based on the LDFZNDA, an adjusted TOPSIS method specific to LDFZN, and an adjusted VIKOR method are put forward in the article. All algorithms are well presented and organized. An example is provided in order to demonstrate the suitability of the suggested methods whereby the dynamics of each technique can be seen with regard to solving the specified problem. It is then accompanied by a sensitivity analysis which is meant to test the effect of alteration in the input parameters on the ranking of alternatives, in order to identify the rank stability and robustness of the models. Lastly, a comparison is performed so as to evaluate the performance of the proposed techniques in relation to the conventional fuzzy decision making techniques. The above comparison shows the benefits of LDFZN framework when managing uncertainty and including degrees of expert confidence. The logical flow in movement through theory to practice makes the article rigorous academically and extremely practical to practice and therefore the article qualifies to be a piece of great contribution within the sphere of fuzzy MCDM. The graphical representation of the structure of this article is given in Figure 1

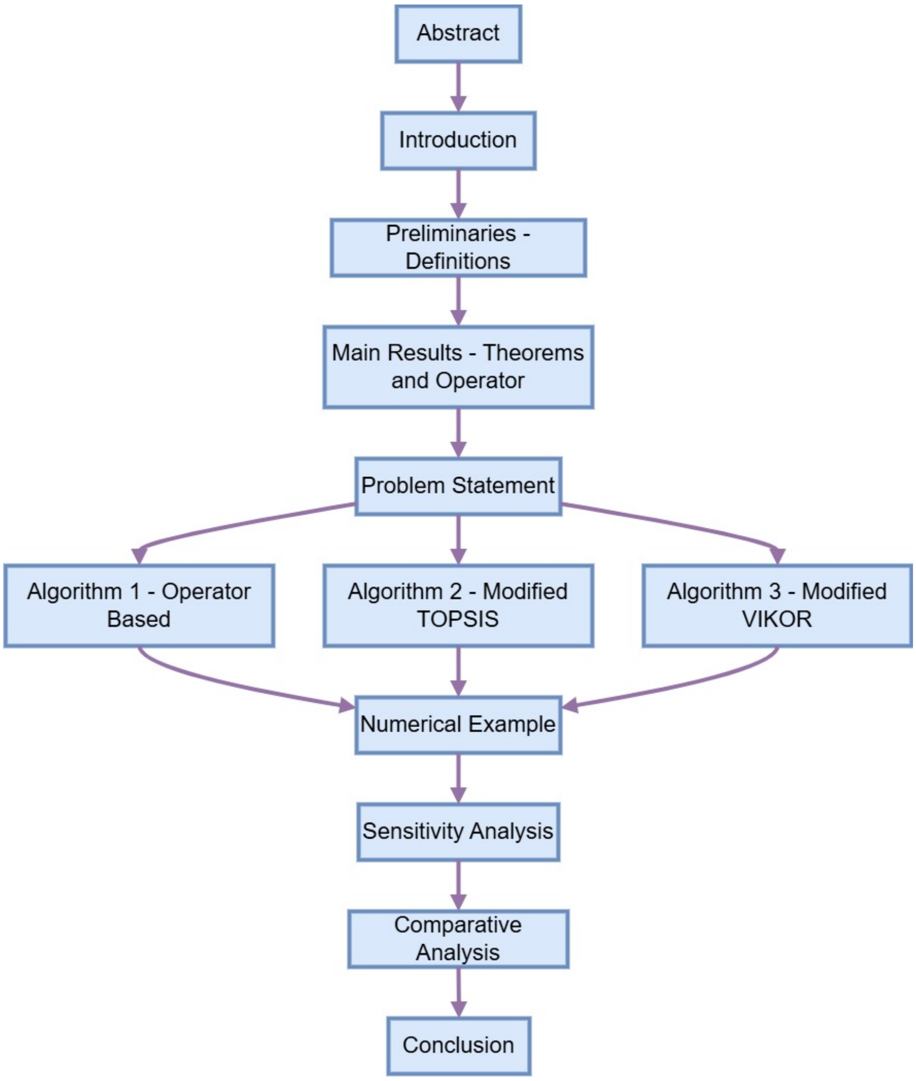

Fig. 1. Ranking from VIKOR.

3. Preliminaries

In this section we present the definitions that are crucial for the further development of this study.

Definition 3.1 (Linear Diophantine Fuzzy Set, $\mathcal{L}$ [23]). Let Y be the universe. A Linear Diophantine Fuzzy Set $\mathcal{L}$ on Y is defined by:

$$\mathcal{L} = \{\tau_j, \langle \mu(\tau_j), \nu(\tau_j) \rangle, \langle \alpha(\tau_j), \gamma(\tau_j) \rangle : \tau_j \in Y\} \tag{1}$$

where $\mu(\tau_j)$, $\nu(\tau_j)$, $\alpha(\tau_j)$, and $\gamma(\tau_j)$ are such that:

$$0 \leq \alpha(\tau_j)\mu(\tau_j) + \gamma(\tau_j)\nu(\tau_j) \leq 1 \quad \forall \tau_j \in Y \tag{2}$$

$$0 \leq \alpha(\tau_j) + \gamma(\tau_j) \leq 1 \tag{3}$$

The hesitation part can be written as:

$$\overline{\mathcal{H}} = 1 - (\alpha(\tau_j)\mu(\tau_j) + \gamma(\tau_j)\nu(\tau_j)) \tag{4}$$

where $\overline{\mathcal{H}}$ is the RP.

The value of $\Im$ can be represented as:

$$\Im = [\langle \mu, \nu \rangle, \langle \alpha, \gamma \rangle] \tag{5}$$

is known as the Linear Diophantine Fuzzy Number (LDFN).

Definition 3.2 (Linear Diophantine Fuzzy Z-numbers Set, $\mathcal{L}^Z$[24])**.** A Linear Diophantine Fuzzy Z-numbers Set $\mathcal{L}^Z$ on universe Y is defined by:

$$\mathcal{L}^Z = \{\tau_j, \langle \mu_\zeta(\tau_j), \mu_\xi(\tau_j), \nu_\zeta(\tau_j), \nu_\xi(\tau_j) \rangle, \langle \alpha_\zeta(\tau_j), \alpha_\xi(\tau_j), \gamma_\zeta(\tau_j), \gamma_\xi(\tau_j) \rangle : \tau_j \in Y\} \tag{6}$$

where $\mu_\zeta(\tau_j), \nu_\zeta(\tau_j), \alpha_\zeta(\tau_j), \gamma_\zeta(\tau_j) \in [0, 1]$ are the membership, non-membership and reference parameters each in turn and they must satisfy:

$$0 \leq \alpha_\zeta(\tau_j)\mu_\zeta(\tau_j) + \gamma_\zeta(\tau_j)\nu_\zeta(\tau_j) \leq 1 \quad \forall \tau_j \in Y \tag{7}$$

$\mu_\xi(\tau_j), \nu_\xi(\tau_j), \alpha_\xi(\tau_j), \gamma_\xi(\tau_j) \in [0, 1]$ are the reliabilities of membership, non-membership and reference parameters respectively, and must satisfy:

$$0 \leq \alpha_\xi(\tau_j)\mu_\xi(\tau_j) + \gamma_\xi(\tau_j)\nu_\xi(\tau_j) \leq 1 \quad \forall \tau_j \in Y \tag{8}$$

The reference parameters must satisfy:

$$0 \leq \alpha_\zeta(\tau_j) + \gamma_\zeta(\tau_j) \leq 1 \tag{9}$$

$$0 \leq \alpha_\xi(\tau_j) + \gamma_\xi(\tau_j) \leq 1 \tag{10}$$

The hesitation part can be written as:

$$\overline{\mathcal{H}} = \frac{2 - (\alpha_\zeta\mu_\zeta + \gamma_\zeta\nu_\zeta + \alpha_\xi\mu_\xi + \gamma_\xi\nu_\xi)}{2} \tag{11}$$

The value of $\Im^Z$ can be represented as:

$$\Im^Z = [\langle \mu_\zeta, \mu_\xi, \nu_\zeta, \nu_\xi \rangle, \langle \alpha_\zeta, \alpha_\xi, \gamma_\zeta, \gamma_\xi \rangle] \tag{12}$$

is known as the Linear Diophantine Fuzzy Z-Number (LDFZN).

Definition 3.3 (Absolute Linear Diophantine Fuzzy Z-number[24])**.** A LDFZN on Ω of the form ${}^1\mathcal{L}^Z_{\mathfrak{D}} = \{(\tau_j, \langle 1, 1, 0, 0 \rangle, \langle 1, 1, 0, 0 \rangle) : \tau_j \in \Omega\}$ is called absolute LDFZN.

Definition 3.4 (Null Linear Diophantine Fuzzy Z-number[24])**.** A LDFZN on Ω of the form ${}^0\mathcal{L}^Z_{\mathfrak{D}} = \{(\tau_j, \langle 0, 0, 1, 1 \rangle, \langle 0, 0, 1, 1 \rangle) : \tau_j \in \Omega\}$ is called empty or null LDFZN.

Definition 3.5 (Score Function of LDFZN $\mathfrak{S}$[24])**.** Let $\Im^Z = (\langle \mu_\zeta, \mu_\xi, \nu_\zeta, \nu_\xi \rangle, \langle \alpha_\zeta, \alpha_\xi, \gamma_\zeta, \gamma_\xi \rangle)$, then the score function $\mathfrak{S}$: LDFZN $\rightarrow [-1, 1]$ is defined by:

$$\mathfrak{S}(\Im^Z) = \frac{1}{2}\left[(\mu_\zeta\mu_\xi - \nu_\zeta\nu_\xi) + (\alpha_\zeta\alpha_\xi - \gamma_\zeta\gamma_\xi)\right] \tag{13}$$

Definition 3.6 (Accuracy Function of LDFZN $\mathfrak{h}$[24])**.** Let $\Im^Z = (\langle \mu_\zeta, \mu_\xi, \nu_\zeta, \nu_\xi \rangle, \langle \alpha_\zeta, \alpha_\xi, \gamma_\zeta, \gamma_\xi \rangle)$, then the accuracy function $\mathfrak{h}$: LDFZN $\rightarrow [0, 1]$ is defined by:

$$\mathfrak{h}(\Im^Z) = \frac{1}{2}\left[\frac{\mu_\zeta\mu_\xi + \nu_\zeta\nu_\xi}{2} + (\alpha_\zeta\alpha_\xi + \gamma_\zeta\gamma_\xi)\right] \tag{14}$$

Definition 3.7 (Operations [25]). Let we have two Fuzzy Znumbers $\tau_1 = (\langle \mu_{\tau_1}, \nu_{\tau_1} \rangle, \langle \alpha_{\tau_1}, \gamma_{\tau_1} \rangle)$ and $\tau_2 = (\langle \mu_{\tau_2}, \nu_{\tau_2} \rangle, \langle \alpha_{\tau_2}, \gamma_{\tau_2} \rangle)$, $\Omega \geq 1, \psi > 0$. Then, Dombi's t-norm and t-conorm operation of Fuzzy Z numbers are defined as follows:

1. $$\tau_1 \oplus \tau_2 = \left\{ \begin{array}{l} \left(1 - \frac{1}{1 + \left[\left(\frac{\mu_{\zeta_{\tau_1}}}{1-\mu_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{\mu_{\zeta_{\tau_2}}}{1-\mu_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\left(\frac{\mu_{\xi_{\tau_1}}}{1-\mu_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{\mu_{\xi_{\tau_2}}}{1-\mu_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\frac{1}{1 + \left[\left(\frac{1-\nu_{\zeta_{\tau_1}}}{\nu_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\nu_{\zeta_{\tau_2}}}{\nu_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\left(\frac{1-\nu_{\xi_{\tau_1}}}{\nu_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\nu_{\xi_{\tau_2}}}{1-\nu_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(1 - \frac{1}{1 + \left[\left(\frac{\alpha_{\zeta_{\tau_1}}}{1-\alpha_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{\alpha_{\zeta_{\tau_2}}}{1-\alpha_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\left(\frac{\alpha_{\xi_{\tau_1}}}{1-\alpha_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{\alpha_{\xi_{\tau_2}}}{1-\alpha_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\frac{1}{1 + \left[\left(\frac{1-\gamma_{\zeta_{\tau_1}}}{\gamma_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\gamma_{\zeta_{\tau_2}}}{\gamma_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\left(\frac{1-\gamma_{\xi_{\tau_1}}}{\gamma_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\gamma_{\xi_{\tau_2}}}{\gamma_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right) \end{array} \right\}.$$

2. $$\tau_1 \otimes \tau_2 = \left\{ \begin{array}{l} \left(\frac{1}{1 + \left[\left(\frac{1-\mu_{\zeta_{\tau_1}}}{\mu_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\mu_{\zeta_{\tau_2}}}{\mu_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\left(\frac{1-\mu_{\xi_{\tau_1}}}{\mu_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\mu_{\xi_{\tau_2}}}{\mu_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(1 - \frac{1}{1 + \left[\left(\frac{\nu_{\zeta_{\tau_1}}}{1-\nu_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{\nu_{\zeta_{\tau_2}}}{1-\nu_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\left(\frac{\nu_{\xi_{\tau_1}}}{1-\nu_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{\nu_{\xi_{\tau_2}}}{1-\nu_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\frac{1}{1 + \left[\left(\frac{1-\alpha_{\zeta_{\tau_1}}}{\alpha_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\alpha_{\zeta_{\tau_2}}}{\alpha_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\left(\frac{1-\alpha_{\xi_{\tau_1}}}{\alpha_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{1-\alpha_{\xi_{\tau_2}}}{\alpha_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(1 - \frac{1}{1 + \left[\left(\frac{\gamma_{\zeta_{\tau_1}}}{1-\gamma_{\zeta_{\tau_1}}} \right)^{\wp} + \left(\frac{\gamma_{\zeta_{\tau_2}}}{1-\gamma_{\zeta_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\left(\frac{\gamma_{\xi_{\tau_1}}}{1-\gamma_{\xi_{\tau_1}}} \right)^{\wp} + \left(\frac{\gamma_{\xi_{\tau_2}}}{1-\gamma_{\xi_{\tau_2}}} \right)^{\wp} \right]^{1/\wp}} \right) \end{array} \right\}.$$

3. $$\psi\tau_1 = \left\{ \begin{array}{l} \left(1-\frac{1}{1+\left[\psi\left(\frac{\mu_{\zeta_{\tau_1}}}{1-\mu_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\psi\left(\frac{\mu_{\xi_{\tau_1}}}{1-\mu_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(\frac{1}{1+\left[\psi\left(\frac{1-\nu_{\zeta_{\tau_1}}}{\nu_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\psi\left(\frac{1-\nu_{\xi_{\tau_1}}}{\nu_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(1-\frac{1}{1+\left[\psi\left(\frac{\alpha_{\zeta_{\tau_1}}}{1-\alpha_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\psi\left(\frac{\alpha_{\xi_{\tau_1}}}{1-\alpha_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(\frac{1}{1+\left[\psi\left(\frac{1-\gamma_{\zeta_{\tau_1}}}{\gamma_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\psi\left(\frac{1-\gamma_{\xi_{\tau_1}}}{\gamma_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right) \end{array} \right\}.$$

4. $$\tau^{\psi} = \left\{ \begin{array}{l} \left(\frac{1}{1+\left[\psi\left(\frac{1-\mu_{\zeta_{\tau_1}}}{\mu_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\psi\left(\frac{1-\mu_{\xi_{\tau_1}}}{\mu_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(1-\frac{1}{1+\left[\psi\left(\frac{\nu_{\zeta_{\tau_1}}}{1-\nu_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\psi\left(\frac{\nu_{\xi_{\tau_1}}}{1-\nu_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(\frac{1}{1+\left[\psi\left(\frac{1-\alpha_{\zeta_{\tau_1}}}{\alpha_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\psi\left(\frac{1-\alpha_{\xi_{\tau_1}}}{\alpha_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(1-\frac{1}{1+\left[\psi\left(\frac{\gamma_{\zeta_{\tau_1}}}{1-\gamma_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\psi\left(\frac{\gamma_{\xi_{\tau_1}}}{1-\gamma_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right) \end{array} \right\}.$$

Propertities: Let τ_1 and τ_2 be two FZNs. Then, we have the following equations:

(1) $\tau_1 \oplus \tau_2 = \tau_2 \oplus \tau_1$;

(2) $\tau_1 \otimes \tau_2 = \tau_2 \otimes \tau_1$;

(3) $\psi(\tau_1 \oplus \tau_2) = \psi\tau_1 \oplus \psi\tau_2, \psi > 0;$

(4) $(\tau_1 \otimes \tau_2)^{\psi} = \tau_1^{\psi} \otimes \tau_2^{\psi};$

(5) $\psi_1\tau \oplus \psi_2\tau = (\psi_1 + \psi_2)\tau;$

(6) $\tau^{\psi_1} \otimes \tau^{\psi_2} = \tau^{(\psi_1+\psi_2)}.$

Definition 3.8 (Linear Diophantine Fuzzy Z-Numbers Dombi Weighted Average (LDFZNDWA))**.** Given a collection of LDFZNs:

$$\tau_j = \Big(\big\langle \mu(\zeta\tau_j), \mu(\xi\tau_j), \nu(\zeta\tau_j), \nu(\xi\tau_j)\big\rangle, \big\langle \alpha(\zeta\tau_j), \alpha(\xi\tau_j), \gamma(\zeta\tau_j), \gamma(\xi\tau_j)\big\rangle\Big), \quad j = 1, 2, \dots, y,$$

The LDFZNDWA operator is defined as:

$$\text{LDFZNDWA}(\tau_1, \tau_2, \dots, \tau_y) = \bigoplus_{j=1}^{y} \omega_j \tau_j$$

Table 1
Benchmarks of some FSs

Fuzzy sets	Researchers	Membership	Non membership	Reference parameters	Reliability
FS	Zadeh [26]	✓	×	×	×
IFS	Atanassov [27]	✓	✓	×	×
PyFS	Yager [28]	✓	✓	×	×
FFS	Senapati and Yager [**Senapati2**, 29]	✓	✓	×	×
q-ROFs	Yager [30]	✓	✓	×	×
LDFS	Riaz and Hashmi [23]	✓	✓	✓	×
LDFZS	Umar and Rukhshanda[24]	✓	✓	✓	✓

4. Main result

Here we will come up with the cumulative value of the offered LDFZNDWA operator

Theorem 4.1. *Let* $\tau_j = \Big(\big\langle \mu(\zeta\tau_j), \mu(\xi\tau_j), \nu(\zeta\tau_j), \nu(\xi\tau_j)\big\rangle, \big\langle \alpha(\zeta\tau_j), \alpha(\xi\tau_j), \gamma(\zeta\tau_j), \gamma(\xi\tau_j)\big\rangle\Big)$, *j = 1, 2, . . . , y, be a collection of LDFZNs. Then, the aggregated value using the LDFZNDWA operator is also an LDFZN and is given by* $\text{LDFZNDWA}(\tau_1, \tau_2, \dots, \tau_y) = \bigoplus_{j=1}^{y} w_j \tau_j$ *where* w_j *represents the weight associated with* τ_j, *satis-*

fying $\sum_{j=1}^{y} w_j = 1$.

$$\left\{\begin{array}{l} \left(1-\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(1-\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right) \end{array}\right\},$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_y)^T$, *with* $\omega_j \geq 0$ *and* $\sum_{j=1}^{y} \omega_j = 1$.

Proof. Using the mathematical induction principle, this theorem is demonstrated. Let $y = 2$. We obtain the following outcome based on the LDFZNDWAs operations

$$\text{LDFZNDWA}(\tau_1, \tau_2) = \tau_1 \oplus \tau_2$$

$$= \left\{\begin{array}{l} \left(1-\frac{1}{1+\left[\left(\frac{\mu_{\zeta_{\tau_1}}}{1-\mu_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\left(\frac{\mu_{\xi_{\tau_1}}}{1-\mu_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(\frac{1}{1+\left[\left(\frac{1-\nu_{\zeta_{\tau_1}}}{\nu_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\left(\frac{1-\nu_{\xi_{\tau_1}}}{\nu_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(1-\frac{1}{1+\left[\left(\frac{\alpha_{\zeta_{\tau_1}}}{1-\alpha_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, 1-\frac{1}{1+\left[\left(\frac{\alpha_{\xi_{\tau_1}}}{1-\alpha_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right), \\ \left(\frac{1}{1+\left[\left(\frac{1-\gamma_{\zeta_{\tau_1}}}{\gamma_{\zeta_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}, \frac{1}{1+\left[\left(\frac{1-\gamma_{\xi_{\tau_1}}}{\gamma_{\xi_{\tau_1}}}\right)^{\wp}\right]^{1/\wp}}\right) \end{array}\right\} \oplus$$

$$
\left\{
\begin{array}{l}
\left(1-\dfrac{1}{1+\left[\left(\frac{\mu_{\zeta_{\tau_2}}}{1-\mu_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}},\ 1-\dfrac{1}{1+\left[\left(\frac{\mu_{\xi_{\tau_2}}}{1-\mu_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right), \\
\left(\dfrac{1}{1+\left[\left(\frac{1-\nu_{\zeta_{\tau_2}}}{\nu_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}},\ \dfrac{1}{1+\left[\left(\frac{1-\nu_{\xi_{\tau_2}}}{\nu_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right), \\
\left(1-\dfrac{1}{1+\left[\left(\frac{\alpha_{\zeta_{\tau_2}}}{1-\alpha_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}},\ 1-\dfrac{1}{1+\left[\left(\frac{\alpha_{\xi_{\tau_2}}}{1-\alpha_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right), \\
\left(\dfrac{1}{1+\left[\left(\frac{1-\gamma_{\zeta_{\tau_2}}}{\gamma_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}},\ \dfrac{1}{1+\left[\left(\frac{1-\gamma_{\xi_{\tau_2}}}{\gamma_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right)
\end{array}
\right\},
$$

$$
= \left\{
\begin{array}{l}
\left(1-\dfrac{1}{1+\left[\left(\frac{\mu_{\zeta_{\tau_1}}}{1-\mu_{\zeta_{\tau_1}}}\right)^{\wp}+\left(\frac{\mu_{\zeta_{\tau_2}}}{1-\mu_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}, 1-\dfrac{1}{1+\left[\left(\frac{\mu_{\xi_{\tau_1}}}{1-\mu_{\xi_{\tau_1}}}\right)^{\wp}+\left(\frac{\mu_{\xi_{\tau_2}}}{1-\mu_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right), \\
\left(\dfrac{1}{1+\left[\left(\frac{1-\nu_{\zeta_{\tau_1}}}{\nu_{\zeta_{\tau_1}}}\right)^{\wp}+\left(\frac{1-\nu_{\zeta_{\tau_2}}}{\nu_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}, \dfrac{1}{1+\left[\left(\frac{1-\nu_{\xi_{\tau_1}}}{\nu_{\xi_{\tau_1}}}\right)^{\wp}+\left(\frac{1-\nu_{\xi_{\tau_2}}}{1-\nu_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right), \\
\left(1-\dfrac{1}{1+\left[\left(\frac{\alpha_{\zeta_{\tau_1}}}{1-\alpha_{\zeta_{\tau_1}}}\right)^{\wp}+\left(\frac{\alpha_{\zeta_{\tau_2}}}{1-\alpha_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}, 1-\dfrac{1}{1+\left[\left(\frac{\alpha_{\xi_{\tau_1}}}{1-\alpha_{\xi_{\tau_1}}}\right)^{\wp}+\left(\frac{\alpha_{\xi_{\tau_2}}}{1-\alpha_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right), \\
\left(\dfrac{1}{1+\left[\left(\frac{1-\gamma_{\zeta_{\tau_1}}}{\gamma_{\zeta_{\tau_1}}}\right)^{\wp}+\left(\frac{1-\gamma_{\zeta_{\tau_2}}}{\gamma_{\zeta_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}, \dfrac{1}{1+\left[\left(\frac{1-\gamma_{\xi_{\tau_1}}}{\gamma_{\xi_{\tau_1}}}\right)^{\wp}+\left(\frac{1-\gamma_{\xi_{\tau_2}}}{\gamma_{\xi_{\tau_2}}}\right)^{\wp}\right]^{1/\wp}}\right)
\end{array}
\right\},
$$

$$
= \left\{ \begin{array}{l}
\left(1 - \frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\
\left(\frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\
\left(1 - \frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\
\left(\frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\sum_{j=1}^{2} \omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right)
\end{array} \right\}.
$$

Now, assume the formula holds for $y = k$

$$
\text{LDFZNDWA}(\tau_1, \tau_2, \ldots, \tau_y) = \bigoplus_{j=1}^{y} \omega_j \tau_j
$$

$$
= \left\{ \begin{array}{l}
\left(1 - \frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\
\left(\frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\
\left(1 - \frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\
\left(\frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right)
\end{array} \right\}.
$$

Then, for $y = k + 1$

$$\text{LDFZNDWA}(\tau_1, \tau_2, \ldots, \tau_y) = \bigoplus_{j=1}^{y} \omega_j \tau_j \oplus \omega_{k+1} \tau_{k+1}$$

$$= \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(1 - \dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \dfrac{1}{1 + \left[\sum_{j=1}^{k} \omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right) \end{array} \right\}$$

$$\oplus \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \left[\omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \dfrac{1}{1 + \left[\omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right) \\ \left(\dfrac{1}{1 + \left[\omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \dfrac{1}{1 + \left[\omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right) \\ \left(1 - \dfrac{1}{1 + \left[\omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \dfrac{1}{1 + \left[\omega_j \left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right) \\ \left(\dfrac{1}{1 + \left[\omega_j \left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \dfrac{1}{1 + \left[\omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right) \end{array} \right\},$$

$$= \left\{ \begin{array}{l} \left(1 - \frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{1-\mu_{\zeta_{\tau_j}}}{\mu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{1-\mu_{\xi_{\tau_j}}}{\mu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(1 - \frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{\gamma_{\zeta_{\tau_j}}}{1-\gamma_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \frac{1}{1 + \left[\sum_{j=1}^{k+1} \omega_j \left(\frac{\gamma_{\xi_{\tau_j}}}{1-\gamma_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right) \end{array} \right\}.$$

□

Theorem 4.2. *Let* $\tau_j = \Big(\big\langle \mu(\zeta\tau_j), \mu(\xi\tau_j), \nu(\zeta\tau_j), \nu(\xi\tau_j) \big\rangle, \big\langle \alpha(\zeta\tau_j), \alpha(\xi\tau_j), \gamma(\zeta\tau_j),$ $\gamma(\xi\tau_j) \big\rangle \Big)$*, be the set of LDFZNs, all identical, where* $j = 1, 2, 3, \ldots, y$*, such that* $\tau_j = \tau$ *for all* j*. Then*

$$\text{LDFZNDWA}(\tau_1, \tau_2, \ldots, \tau_y) = \tau.$$

Proof. Since

$$\tau_j = (\langle \mu(\zeta\tau_j),\ \mu(\xi\tau_j),\ \nu(\zeta\tau_j),\ \nu(\xi\tau_j) \rangle\ ,\ \langle \alpha(\zeta\tau_j),\ \alpha(\xi\tau_j),\ \gamma(\zeta\tau_j),\ \gamma(\xi\tau_j) \rangle) = \tau,$$

for all j = 1, 2, ..., y.
Then we have

$$\text{LDFZNDWA}(\tau_1, \tau_2, \ldots, \tau_y) = \bigoplus_{j=1}^{y} \omega_j \tau_j$$

$$= \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, 1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right), \\ \left(\dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}, \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \right) \end{array} \right\},$$

$$= \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}} \right)^{\wp}}, 1 - \dfrac{1}{1 + \omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}} \right)^{\wp}} \right), \\ \left(\dfrac{1}{1 + \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp}}, \dfrac{1}{1 + \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp}} \right), \\ \left(1 - \dfrac{1}{1 + \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp}}, 1 - \dfrac{1}{1 + \omega_j \left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}} \right)^{\wp}} \right), \\ \left(\dfrac{1}{1 + \omega_j \left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}} \right)^{\wp}}, \dfrac{1}{1 + \omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}} \right)^{\wp}} \right) \end{array} \right\},$$

$$= \left\{ \begin{array}{l} \left(1 - \frac{1}{1+\omega_j \left(\frac{\mu_{\zeta\tau_j}}{1-\mu_{\zeta\tau_j}} \right)}, 1 - \frac{1}{1+\omega_j \left(\frac{\mu_{\xi\tau_j}}{1-\mu_{\xi\tau_j}} \right)} \right), \\ \left(\frac{1}{1+\omega_j \left(\frac{1-\nu_{\zeta\tau_j}}{\nu_{\zeta\tau_j}} \right)}, \frac{1}{1+\omega_j \left(\frac{1-\nu_{\xi\tau_j}}{\nu_{\xi\tau_j}} \right)} \right), \\ \left(1 - \frac{1}{1+\omega_j \left(\frac{\alpha_{\zeta\tau_j}}{1-\alpha_{\zeta\tau_j}} \right)}, 1 - \frac{1}{1+\omega_j \left(\frac{\alpha_{\xi\tau_j}}{1-\alpha_{\xi\tau_j}} \right)} \right), \\ \left(\frac{1}{1+\omega_j \left(\frac{1-\gamma_{\zeta\tau_j}}{\gamma_{\zeta\tau_j}} \right)}, \frac{1}{1+\omega_j \left(\frac{1-\gamma_{\xi\tau_j}}{\gamma_{\xi\tau_j}} \right)} \right) \end{array} \right\},$$

$$= \left(\langle \mu(\zeta\tau_j),\, \mu(\xi\tau_j),\, \nu(\zeta\tau_j),\, \nu(\xi\tau_j) \rangle,\, \langle \alpha(\zeta\tau_j),\, \alpha(\xi\tau_j),\, \gamma(\zeta\tau_j),\, \gamma(\xi\tau_j) \rangle \right) = \tau.$$

□

Theorem 4.3. *Suppose* $\tau_j = \Big(\langle \mu(\zeta\tau_j), \mu(\xi\tau_j), \nu(\zeta\tau_j), \nu(\xi\tau_j) \rangle, \langle \alpha(\zeta\tau_j), \alpha(\xi\tau_j), \gamma(\zeta\tau_j), \gamma(\xi\tau_j) \rangle \Big)$, *for* $j = 1, 2, \ldots, y$, *be a set of LDFZNs, and define* $\tau^- = \min(\tau_1, \tau_2, \ldots, \tau_y)$ *and* $\tau^+ = \max(\tau_1, \tau_2, \ldots, \tau_y)$. *Then,* $\tau^- \le \mathrm{LDFZNs}_\omega(\tau_1, \tau_2, \ldots, \tau_y) \le \tau^+$.

Proof. Let $\tau_j = \left(\langle \mu(\zeta\tau_j), \mu(\xi\tau_j), \nu(\zeta\tau_j), \nu(\xi\tau_j) \rangle, \langle \alpha(\zeta\tau_j), \alpha(\xi\tau_j), \gamma(\zeta\tau_j), \gamma(\xi\tau_j) \rangle \right), j = 1, 2, \ldots, y$, be a set of LDFZNs. Define $\tau^- = \left(\langle \mu^-(\zeta\tau_j), \mu^-(\xi\tau_j), \nu^-(\zeta\tau_j), \nu^-(\xi\tau_j) \rangle, \langle \alpha^-(\zeta\tau_j), \alpha^-(\xi\tau_j), \gamma^-(\zeta\tau_j), \gamma^-(\xi\tau_j) \rangle \right)$, and $\tau^+ = \left(\langle \mu^+(\zeta\tau_j), \mu^+(\xi\tau_j), \nu^+(\zeta\tau_j), \nu^+(\xi\tau_j) \rangle, \langle \alpha^+(\zeta\tau_j), \alpha^+(\xi\tau_j), \gamma^+(\zeta\tau_j), \gamma^+(\xi\tau_j) \rangle \right)$. The ordering $\mu^-_{\zeta_{\tau_j}} \le \mu_{\zeta_{\tau_j}} \le \mu^+_{\zeta_{\tau_j}}$ is preserved by the weighted generalized aggregation function, yielding

$$1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu^-_{\zeta_{\tau_j}}}{1-\mu^-_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le 1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le 1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu^+_{\zeta_{\tau_j}}}{1+\mu^+_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}.$$

Similarly for all the remaining components of τ_j.We compare the values of τ^- and τ^+

$$\frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu^-_{\zeta_{\tau_j}}}{\nu^-_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{1+\nu^+_{\zeta_{\tau_j}}}{\nu^+_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}.$$

$$1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu^-_{\xi_{\tau_j}}}{1-\mu^-_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le 1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le 1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu^+_{\xi_{\tau_j}}}{1-\mu^+_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}.$$

$$\frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu^-_{\xi_{\tau_j}}}{\nu^-_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu^+_{\xi_{\tau_j}}}{\nu^+_{\xi_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}.$$

$$1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\alpha^-_{\zeta_{\tau_j}}}{1-\alpha^-_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le 1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}} \le 1 - \frac{1}{1+\left[\sum_{j=1}^{y} \omega_j \left(\frac{\alpha^+_{\zeta_{\tau_j}}}{1-\alpha^+_{\zeta_{\tau_j}}} \right)^{\wp} \right]^{1/\wp}}.$$

$$\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma^-_{\zeta\tau_j}}{\gamma^-_{\zeta\tau_j}}\right)^{\wp}\right]^{1/\wp}} \leq \frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma_{\zeta\tau_j}}{\gamma_{\zeta\tau_j}}\right)^{\wp}\right]^{1/\wp}} \leq \frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma^+_{\zeta\tau_j}}{\gamma^+_{\zeta\tau_j}}\right)^{\wp}\right]^{1/\wp}}.$$
$$1-\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha^-_{\xi\tau_j}}{1-\alpha^-_{\xi\tau_j}}\right)^{\wp}\right]^{1/\wp}} \leq 1-\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha_{\xi\tau_j}}{1-\alpha_{\xi\tau_j}}\right)^{\wp}\right]^{1/\wp}} \leq 1-\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha^+_{\xi\tau_j}}{1-\alpha^+_{\xi\tau_j}}\right)^{\wp}\right]^{1/\wp}}.$$
$$\frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma^-_{\xi\tau_j}}{\gamma^-_{\xi\tau_j}}\right)^{\wp}\right]^{1/\wp}} \leq \frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma_{\xi\tau_j}}{\gamma_{\xi\tau_j}}\right)^{\wp}\right]^{1/\wp}} \leq \frac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma^+_{\xi\tau_j}}{\gamma^+_{\xi\tau_j}}\right)^{\wp}\right]^{1/\wp}}.$$

Since all components are between the corresponding components of τ^- and τ^+, we have the following

$$\tau^- \leq LDFZNs_\omega(\tau_1,\tau_2,\ldots,\tau_y) \leq \tau^+.$$

□

Theorem 4.4. *Let* $\tau_j = (\langle\mu(\zeta\tau_j),\mu(\xi\tau_j),\nu(\zeta\tau_j),\nu(\xi\tau_j)\rangle, \langle\alpha(\zeta\tau_j),\alpha(\xi\tau_j),\gamma(\zeta\tau_j),\gamma(\xi\tau_j)\rangle)$ *, and* $\tau_j^* = (\langle\mu^*(\zeta\tau_j),\mu^*(\xi\tau_j),\nu^*(\zeta\tau_j),\nu^*(\xi\tau_j)\rangle, \langle\alpha^*(\zeta\tau_j),\alpha^*(\xi\tau_j),\gamma^*(\zeta\tau_j),\gamma^*(\xi\tau_j)\rangle), j=1,2,3,\ldots,y.$ *Then* $LDFZNs_\omega(\tau_1,\tau_2,\tau_3,\ldots,\tau_y) \leq LDFZNs_\omega(\tau_1^*,\tau_2^*,\tau_3^*,\ldots,\tau_y^*)$.

Proof. Since

$$\begin{aligned}
&\mu(\zeta\tau_j)\leq\mu^*(\zeta\tau_j),\quad \mu(\xi\tau_j)\leq\mu^*(\xi\tau_j),\quad \nu(\zeta\tau_j)\leq\nu^*(\zeta\tau_j),\quad \nu(\xi\tau_j)\leq\nu^*(\xi\tau_j),\\
&\alpha(\zeta\tau_j)\leq\alpha^*(\zeta\tau_j),\quad \alpha(\xi\tau_j)\leq\alpha^*(\xi\tau_j),\quad \gamma(\zeta\tau_j)\leq\gamma^*(\zeta\tau_j),\quad \gamma(\xi\tau_j)\leq\gamma^*(\xi\tau_j),
\end{aligned}$$

we have

$$\begin{aligned}
&1-\mu^*(\zeta\tau_j)\leq 1-\mu(\zeta\tau_j),\\
&1-\prod_{j=1}^{y}(1-\mu^*(\zeta\tau_j))^{\omega}\leq 1-\prod_{j=1}^{y}(1-\mu(\zeta\tau_j))^{\omega},\\
&\implies \left(1-\prod_{j=1}^{y}(1-\mu^*(\zeta\tau_j))^{\omega}\right)\leq\left(1-\prod_{j=1}^{y}(1-\mu(\zeta\tau_j))^{\omega}\right).
\end{aligned}$$

$$\begin{aligned}
&1-\mu^*(\xi\tau_j)\leq 1-\mu(\xi\tau_j),\\
&\prod_{j=1}^{y}(1-\mu^*(\xi\tau_j))^{\omega}\leq\prod_{j=1}^{y}(1-\mu(\xi\tau_j))^{\omega},\\
&\implies \left(1-\prod_{j=1}^{y}(1-\mu^*(\xi\tau_j))^{\omega}\leq 1-\prod_{j=1}^{y}(1-\mu(\xi\tau_j))^{\omega}\right).
\end{aligned}$$

$$\begin{aligned}
&1-\nu^*(\zeta\tau_j)\leq 1-\nu(\zeta\tau_j),\\
&\prod_{j=1}^{y}(1-\nu^*(\zeta\tau_j))^{\omega}\leq\prod_{j=1}^{y}(1-\nu(\zeta\tau_j))^{\omega},\\
&\implies \left(1-\prod_{j=1}^{y}(1-\nu^*(\zeta\tau_j))^{\omega}\leq 1-\prod_{j=1}^{y}(1-\nu(\zeta\tau_j))^{\omega}\right).
\end{aligned}$$

$$1-\nu^*(\xi\tau_j)\leq 1-\nu(\xi\tau_j),$$
$$\prod_{j=1}^{y}(1-\nu^*(\xi\tau_j))^{\omega}\leq\prod_{j=1}^{y}(1-\nu(\xi\tau_j))^{\omega},$$
$$\implies\Big(1-\prod_{j=1}^{y}(1-\nu^*(\xi\tau_j))^{\omega}\leq 1-\prod_{j=1}^{y}(1-\nu(\xi\tau_j))^{\omega}\Big).$$

Similarly, we can show the following

$$\alpha(\zeta\tau_j)\leq\alpha^*(\zeta\tau_j),\quad \alpha(\xi\tau_j)\leq\alpha^*(\xi\tau_j),\gamma(\zeta\tau_j)\leq\gamma^*(\zeta\tau_j),\quad \gamma(\xi\tau_j)\leq\gamma^*(\xi\tau_j).$$

$$\left\{\begin{array}{l}\left(1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right),\\ \left(\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right),\\ \left(1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right),\\ \left(\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right)\end{array}\right\}\leq$$

$$\left\{\begin{array}{l}\left(1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\mu^*_{\zeta_{\tau_j}}}{1-\mu^*_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\mu^*_{\xi_{\tau_j}}}{1-\mu^*_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right),\\ \left(\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\nu^*_{\zeta_{\tau_j}}}{\nu^*_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\nu^*_{\xi_{\tau_j}}}{\nu^*_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right),\\ \left(1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha^*_{\zeta_{\tau_j}}}{1-\alpha^*_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},1-\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{\alpha^*_{\xi_{\tau_j}}}{1-\alpha^*_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right),\\ \left(\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma^*_{\zeta_{\tau_j}}}{\gamma^*_{\zeta_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}},\dfrac{1}{1+\left[\sum_{j=1}^{y}\omega_j\left(\frac{1-\gamma^*_{\xi_{\tau_j}}}{\gamma^*_{\xi_{\tau_j}}}\right)^{\wp}\right]^{1/\wp}}\right)\end{array}\right\}.$$

□

Theorem 4.5. *Let* $\tau_j = \left(\left\langle \mu(\zeta\tau_j),\, \mu(\xi\tau_j),\, \nu(\zeta\tau_j),\, \nu(\xi\tau_j)\right\rangle,\, \left\langle \alpha(\zeta\tau_j),\, \alpha(\xi\tau_j),\, \gamma(\zeta\tau_j),\, \gamma(\xi\tau_j)\right\rangle\right)$, *j = 1, 2, ..., y, be a set of linear Diophantine fuzzy Z number ordered weighted averaging (LDFC-DOWA). Then, the LDFCDOWA operator of dimension* y *is a function LDFCDOWA* : $\tau^y \to \tau$, *such that LDFCDOWA*$(\tau_1, \tau_2, \ldots, \tau_y) = \bigoplus_{j=1}^{y} \omega\tau_j$.

$$\bigoplus_{j=1}^{y} \omega\tau_j = \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{\mu(\zeta\tau_j)}{1-\mu(\zeta\tau_j)} \right)^{\wp} \right)^{1/\wp}}, 1 - \dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{\mu(\xi\tau_j)}{1-\mu(\xi\tau_j)} \right)^{\wp} \right)^{1/\wp}} \right), \\ \left(\dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{1-\nu(\zeta\tau_j)}{\nu(\zeta\tau_j)} \right)^{\wp} \right)^{1/\wp}}, \dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{1-\nu(\xi\tau_j)}{\nu(\xi\tau_j)} \right)^{\wp} \right)^{1/\wp}} \right), \\ \left(1 - \dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{\alpha(\zeta\tau_j)}{1-\alpha(\zeta\tau_j)} \right)^{\wp} \right)^{1/\wp}}, 1 - \dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{\alpha(\xi\tau_j)}{1-\alpha(\xi\tau_j)} \right)^{\wp} \right)^{1/\wp}} \right), \\ \left(\dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{1-\gamma(\zeta\tau_j)}{\gamma(\zeta\tau_j)} \right)^{\wp} \right)^{1/\wp}}, \dfrac{1}{1 + \left(\sum\limits_{j=1}^{y} \omega_j \left(\frac{1-\gamma(\xi\tau_j)}{\gamma(\xi\tau_j)} \right)^{\wp} \right)^{1/\wp}} \right) \end{array} \right\}.$$

Here $\omega = (\omega_1, \omega_2, \ldots, \omega_y)^T$ *corresponds to* τ_j $(j = 1, 2, 3, \ldots, y)$ *with* $\omega_j \geq 0$ *and* $\sum_{j=1}^{y} \omega_j = 1$.

Proof. The proof is obvious. □

Theorem 4.6. *(Idempotency) Let*

$$\tau_j = \left(\left\langle \mu(\zeta\tau_j),\, \mu(\xi\tau_j),\, \nu(\zeta\tau_j),\, \nu(\xi\tau_j)\right\rangle,\, \left\langle \alpha(\zeta\tau_j),\, \alpha(\xi\tau_j),\, \gamma(\zeta\tau_j),\, \gamma(\xi\tau_j)\right\rangle\right),$$

, are identical, i.e.,$\tau_j = \tau$, *for all* j.*Then,*

$$\text{LDFZNOWA}(\tau_1, \tau_2, \ldots, \tau_y) = \tau.$$

Proof. Proof is obvious. □

Theorem 4.7. *(Boundedness)*
Suppose

$$\tau_j = \left(\left\langle \mu(\zeta\tau_j),\, \mu(\xi\tau_j),\, \nu(\zeta\tau_j),\, \nu(\xi\tau_j)\right\rangle,\, \left\langle \alpha(\zeta\tau_j),\, \alpha(\xi\tau_j),\, \gamma(\zeta\tau_j),\, \gamma(\xi\tau_j)\right\rangle\right), \quad j = 1, 2, 3, \ldots, y,$$

be a set of LDFZNOWA, and define

$$\tau^- = \min(\tau_1, \tau_2, \tau_3, \ldots, \tau_y), \quad \tau^+ = \max(\tau_1, \tau_2, \tau_3, \ldots, \tau_y).$$

Then,

$$\tau^- \leq LDFZNOWAs_{\omega}(\tau_1, \tau_2, \tau_3, \ldots, \tau_y) \leq \tau^+.$$

Proof. Proof is obvious. □

Theorem 4.8. *(Monotonicity) Let*

$$\tau_j = (\langle \mu(\zeta\tau_j),\, \mu(\xi\tau_j),\, \nu(\zeta\tau_j),\, \nu(\xi\tau_j)\rangle\,,\ \langle \alpha(\zeta\tau_j),\, \alpha(\xi\tau_j),\, \gamma(\zeta\tau_j),\, \gamma(\xi\tau_j)\rangle)$$

be a number of LDFZNs,if$\tau_j \leq \tau_j^*$ *Then,*

$$LDFZOWA_{\omega}(\tau_1, \tau_2, \tau_3, \ldots, \tau_y) \leq LDFZNOWA_{\omega}(\tau_1^*, \tau_2^*, \tau_3^*, \ldots, \tau_y^*).$$

Proof. Proof is obvious. □

Theorem 4.9. *Let* $\tau_j = \Big(\big\langle \mu(\zeta\tau_j),\, \mu(\xi\tau_j),\, \nu(\zeta\tau_j),\, \nu(\xi\tau_j)\big\rangle,\, \big\langle \alpha(\zeta\tau_j),\, \alpha(\xi\tau_j),\, \gamma(\zeta\tau_j),\, \gamma(\xi\tau_j)\big\rangle\Big), \quad j = 1, 2, 3, \ldots, y,$ *be the set of LDFZNs. Then, the aggregated value using the LDFZNDWA operator is also an LDFZN, defined as:*

$$\text{LDFZNDWA}(\tau_1, \tau_2, \ldots, \tau_y) = \bigoplus_{j=1}^{y} \tau_j =$$

$$\left\{\begin{array}{l}
\left(1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu_{\zeta_{\tau_j}}}{1-\mu_{\zeta_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}, 1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\mu_{\xi_{\tau_j}}}{1-\mu_{\xi_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}\right) \\
\left(\dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu_{\zeta_{\tau_j}}}{\nu_{\zeta_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}, \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\nu_{\xi_{\tau_j}}}{\nu_{\xi_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}\right) \\
\left(1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\alpha_{\zeta_{\tau_j}}}{1-\alpha_{\zeta_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}, 1 - \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{\alpha_{\xi_{\tau_j}}}{1-\alpha_{\xi_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}\right) \\
\left(\dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\gamma_{\zeta_{\tau_j}}}{\gamma_{\zeta_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}, \dfrac{1}{1 + \left[\sum_{j=1}^{y} \omega_j \left(\frac{1-\gamma_{\xi_{\tau_j}}}{\gamma_{\xi_{\tau_j}}}\right)^{\Omega}\right]^{1/\Omega}}\right)
\end{array}\right\}$$

$\omega = (\omega_1, \omega_2, \ldots, \omega_y)^T$ *where* $\omega_j \geq 0$ *and* $\sum_{j=1}^{y} \omega_j = 1$

Proof. Proof is obvious. □

5. Problem statement

Pakistan is facing a serious urban transportation crisis, especially in major cities like Karachi, Lahore, and Islamabad, where mobility issues are becoming a real threat to both economic productivity and the overall quality of life. With an urbanization rate nearing 3 percent each year—one of the highest in South Asia—our infrastructure just can't keep up. This situation has resulted in persistent traffic congestion, incurring an economic cost of approximately Rs. 1 trillion annually due to diminished productivity. The complexity of this scenario poses a multifaceted planning challenge for policymakers who are tasked with achieving a equilibrium between: Financial Constraints: Given the limitations of public budgets alongside various competing development priorities, any proposed transportation solutions must demonstrate robust cost-benefit analyses. The construction of metro systems is notably capital-intensive, with expenditures ranging from 50 to 100 million per kilometer, thereby exerting considerable pressure on financial resources. Conversely, more economical alternatives such as Bus Rapid Transit (BRT), which costs between 5 and 20 million per kilometer, may not adequately address the demand for capacity. Urban Mobility Requirements: In urban centers where population densities surpass 30,000 individuals per square kilometer, the strain on transportation infrastructures is substantial. Traditional interventions, such as road widening, have proven ineffective due to the phenomenon of induced demand, and incremental solutions often fail to address the overarching issue of systemic connectivity. Environmental Imperatives: Pakistan is particularly vulnerable to climate change, as seen in devastating floods and severe air pollution (with Lahore being named the world's most polluted city in 2023). This situation calls for low-carbon transport solutions, especially since the transportation sector is responsible for nearly 40 percent of urban air pollution and 15 percent of national greenhouse gas emissions. Implementation Challenges: Acquiring right-of-way, relocating utilities, and coordinating among various levels of government create significant obstacles. For instance, Lahore's Orange Line Metro faced years of delays due to disputes over heritage preservation, while Karachi's Green Line BRT struggled with land acquisition and funding hurdles. The limitations of traditional planning tools become especially clear when we cconsider the following.the following. The gap between short-term political agendas and the long-term benefits of infrastructure is a real challenge. We also face difficulties in measuring how different transport options impact social equity. There are uncertainties about how urban growth will unfold in the future and how technology might disrupt our plans. We have to weigh the pros and cons of taking a comprehensive approach to systems versus making gradual improvements. This study uses a fuzzy methodology to tackle these complexities by: Recognizing that transportation alternatives can have both significant advantages and serious downsides at the same time. Bringing together expert opinions with hard data. Acknowledging the uncertainty that comes with long-term infrastructure planning. Allowing us to compare very different types of solutions—like capital-heavy rail systems versus behavioral changes such as non-motorized transport—within a unified analytical framework. The urgency of this analysis is highlighted by Pakistan's commitments to the Paris Agreement, which aims for a 35 percent reduction in greenhouse gas emissions by 2030, and the UN Sustainable Development Goals, particularly SDG 11.2 focused on sustainable transport. This is compounded by the fact that our current transportation systems are already stretched to their limits during peak hours. Without a systematic, data-driven approach to planning, cities risk: Underinvesting in mobility solutions, which leads to ongoing congestion and pollution. Overinvesting in unsuitable technologies that become financial burdens. Chasing fragmented projects that don't create cohesive networks. This research equips policymakers with a solid decision-support tool that takes into account Pakistan's unique urban landscape—characterized by high levels of informality (with 30-50 percent of urban trips made via para-transit), energy limitations, and governance hurdles—while providing a structured way to assess alternatives against various competing goals.the following.the following.the following.the following.

Decision Criteria

Cost-effectiveness $(\mathcal{G}_1)$: Outlining the first financial commitment or set up along with future servicing costs
Traffic Decongestion Potential $(\mathcal{G}_2)$ Ability to alleviate citywide congestion
Environmental Impact $(\mathcal{G}_3)$: Scope of emission cuts and ecological effects including carbon footprint
Implementation Feasibility $(\mathcal{G}_4)$ Existing infrastructure and how readily the new system can be integrated

Transportation Alternatives

Metro Rail System (λ_1): High capacity electric rail network
Bus Rapid Transit (λ_2): Bus system with stations and exclusive lane use
Electric Vehicle Infrastructure (λ_3) Network of charging stations with supporting policies
Non-Motorized Transport (λ_4) Pedestrian infrastructure and cycling facilities

6. MCDM algorithm using linear Diophantine fuzzy Z-numbers

Algorithm 1 LDFZN-Based Multi-Criteria Decision Making

1. **Step 1: Define the Decision Problem**
   (a) **Identify Alternatives**:
   - Let there be m alternatives $\lambda_1, \lambda_2, \ldots, \lambda_m$ to be evaluated.

   (b) **Define Criteria**:
   - Let there be n criteria $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$. With deep understanding of the problem, choice of suitable criteria is very important.

   (c) **Construct the Decision Matrix**:
   - Each alternative λ_i is evaluated under each criterion $\mathcal{G}_j$ using a LDFZN.

2. **Step 2: Aggregation Using LDFZNWA Operator**
   (a) **Assign Weights to Criteria (if applicable)**:
   - Let ω_j be the weight for criterion $\mathcal{G}_j$, where $\sum_{j=1}^{n} \omega_j = 1$

   (b) **Apply the LDFZNWA Operator**:
   - For each alternative λ_i, compute the aggregated LDFZN using LDFZNDWA

3. **Step 3: Compute Score Values**
   (a) **Apply Score Function**:

$$SF(\mathcal{Z}) = \frac{1}{2}\left[\left((\mu(\zeta\tau_j)\cdot\mu(\xi\tau_j))-(\nu(\zeta\tau_j)\cdot\nu(\xi\tau_j))\right)+\left((\alpha(\zeta\tau_j)\cdot\alpha(\xi\tau_j)-(\gamma(\zeta\tau_j)\cdot\gamma(\xi\tau_j))\right)\right]$$

4. **Step 4: Rank Alternatives and Select the Best Option**
   (a) **Rank in Descending Order**:
   - Sort alternatives based on their scores:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m$$

   where λ_1 has the highest score.

   (b) **Final Decision**:
   - The alternative with the highest score is the optimal choice.

Numerical Example

Step 1: Make Judgment Matrices

Create judgment matrices from the decision-makers' assessments (see Table 1). The input from different professionals is represented by these matrices. Each alternative (λ_1, λ_2, λ_3, λ_4,) is evaluated in the decision-maker's assessment matrix ($\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$) is displayed in Table 1. The values show the relative performance of each alternative for each decision maker.

Table 2
Linear Diophantine Fuzzy Z-Numbers given by Decision Makers

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
λ_1	(0.9, 0.6),(0.2, 0.2),(0.9, 0.9),(0.3, 0.2)	(0.8, 0.7),(0.4, 0.3),(0.9, 0.9),(0.2, 0.1)	(0.7, 0.6),(0.1, 0.1),(0.8, 0.8),(0.1, 0.6)	(0.6, 0.6),(0.9, 0.2),(0.4, 0.8),(0.7, 0.1)
λ_2	(0.7, 0.6),(0.4, 0.3),(0.9, 0.8),(0.2, 0.1)	(0.6, 0.9),(0.3, 0.2),(0.9, 0.9),(0.1, 0.1)	(0.9, 0.8),(0.1, 0.5),(0.7, 0.6),(0.4, 0.3)	(0.8, 0.7),(0.5, 0.4),(0.6, 0.9),(0.1, 0.1)
λ_3	(0.8, 0.7),(0.1, 0.5),(0.6, 0.9),(0.4, 0.3)	(0.9, 0.8),(0.2, 0.1),(0.7, 0.6),(0.5, 0.4)	(0.7, 0.6),(0.5, 0.4),(0.9, 0.8),(0.1, 0.1)	(0.6, 0.9),(0.4, 0.3),(0.8, 0.7),(0.2, 0.1)
λ_4	(0.7, 0.6),(0.1, 0.5),(0.9, 0.8),(0.4, 0.3)	(0.6, 0.9),(0.5, 0.4),(0.8, 0.7),(0.3, 0.2)	(0.9, 0.8),(0.3, 0.2),(0.7, 0.6),(0.2, 0.5)	(0.8, 0.7),(0.5, 0.5),(0.6, 0.9),(0.5, 0.5)

Step 2: Utilize the operator for the Linear Diaphatine Fuzzy Number Weighted Average Operator (LD-FZNWA):
Utilize the mathematical formula for the LDFZNWA aggregation operator. The overall performance ratings of four options across various sub-attributes are calculated in Table 8 by summing the assessments of the four decision-makers ($\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$,$\mathcal{G}_5$). All the weights are considered 2.5 .

The values provide the total evaluation of each choice according to several criteria.

Table 3
Apply LDFZNWA operator on Table 1

Alternatives	Aggregated value of LD-FNS
λ_1	(0.8868, 0.8868), (0.1132, 0.1132), (0.8936, 0.8936), (0.1064, 0.1064)
λ_2	(0.8868, 0.8868), (0.1132, 0.2230), (0.8936, 0.8936), (0.1064, 0.1026)
λ_3	(0.8868, 0.8868), (0.1132, 0.1132), (0.8868, 0.8868), (0.1132, 0.1064)
λ_4	(0.8868, 0.8868), (0.1132, 0.2231), (0.8868, 0.8868), (0.2230, 0.2230)
λ_5	(0.8868, 0.8868), (0.1132, 0.1132), (0.8936, 0.8936), (0.1064, 0.1064)

Step 3. Determine the Score Values
Utilizing the aggregation findings, determine the score values for each choice.
To rank alternatives, compare the score values.
Find the score value using the formula:

$$\lambda_1 = 0.6604 \quad \lambda_2 = 0.6439 \quad \lambda_3 = 0.6140 \quad \lambda_4 = 0.5622$$

Step 4. Making Decisions and Ranking
Based on the determined scores, order the options.
Ultimately, choose the one that has the highest ranking. The ranking of the alternatives is presented in Figure 2.

$$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$$

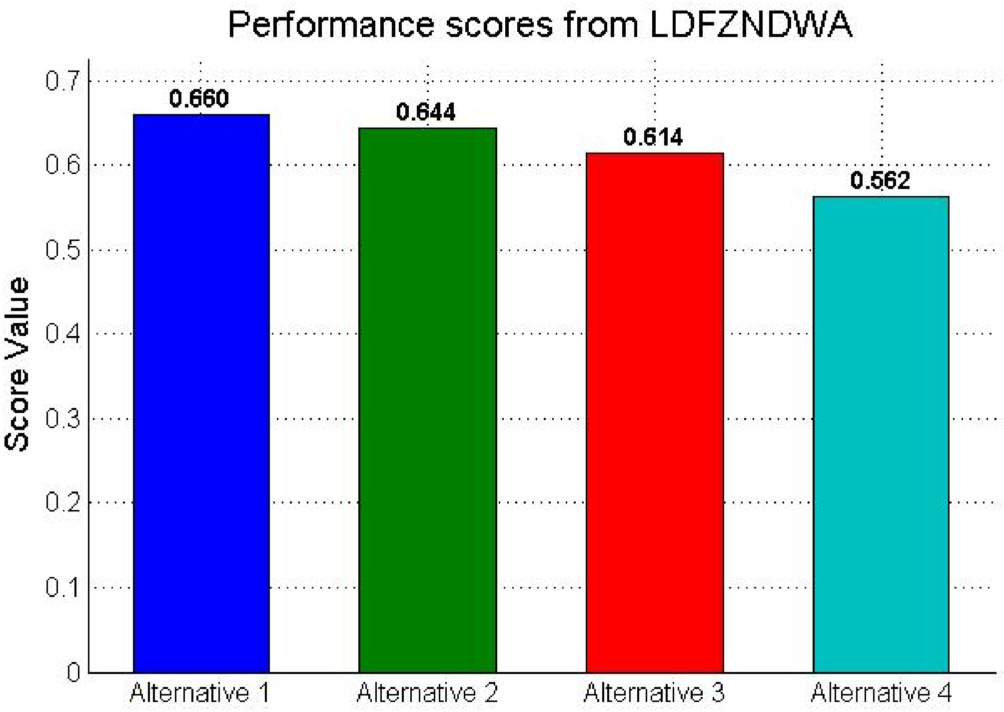

Fig. 2. Score of alternatives obtained using LDFZNDWA Operator .

7. TOPSIS algorithm

TOPSIS Method on Numerical Problem

Step 1: Obtain crisp decision matrix:

By applying the score function on each entry of Table 2 we get Table 4.

Table 4
Crisp matrix obtained by applying score function on LDFZN

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
λ_1	0.6250	0.6150	0.4950	0.5050
λ_2	0.5000	0.6400	0.4850	0.4450
λ_3	0.4650	0.4600	0.4650	0.4800
λ_4	0.4850	0.4200	0.4900	0.3000

Step 1: Normalize the Decision Matrix

After normalization we get Table 5

Table 5
Normalized Decision matrix

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
λ_1	0.5981	0.5672	0.5115	0.5742
λ_2	0.4785	0.5902	0.5012	0.5060
λ_3	0.4450	0.4242	0.4805	0.5458
λ_4	0.4641	0.3873	0.5063	0.3411

Algorithm 2 TOPSIS Algorithm

Input: Decision matrix based on LDFZN $D = [x_{ij}]$ of size $m \times n$, weights vector $\omega = [\omega_1, \omega_2, \ldots, \omega_y]$
Output: Ranking of alternatives

1. **Obtain crisp decision matrix:**
   Convert LDFZN based decision matrix into crisp matrix by applying score function on each LDFZN.

$$SF(\mathcal{Z}) = \frac{1}{2}\left[\left((\mu(\zeta\tau_j) \cdot \mu(\xi\tau_j)) - (\nu(\zeta\tau_j) \cdot \nu(\xi\tau_j))\right) + \left((\alpha(\zeta\tau_j) \cdot \alpha(\xi\tau_j) - (\gamma(\zeta\tau_j) \cdot \gamma(\xi\tau_j))\right)\right]$$

2. **Normalize the decision matrix:**

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{y} x_{ij}^2}}, \quad \text{for } i = 1, \ldots, m;\ j = 1, \ldots, n$$

3. **Construct the weighted normalized decision matrix:**

$$v_{ij} = \omega_j \cdot r_{ij}, \quad \text{for } i = 1, \ldots, m;\ j = 1, \ldots, n$$

4. **Determine the positive ideal (λ^+) and negative ideal (λ^-) solutions:**

$$\lambda^+ = \left\{\max_i v_{ij} \mid j \in J_{\text{benefit}};\ \min_i v_{ij} \mid j \in J_{\text{cost}}\right\}$$

$$\lambda^+ = \left\{\min_i v_{ij} \mid j \in J_{\text{benefit}};\ \max_i v_{ij} \mid j \in J_{\text{cost}}\right\}$$

5. **Calculate the separation measures:**

$$S_i^+ = \sqrt{\sum_{j=1}^{y}(v_{ij} - \lambda_j^+)^2}, \quad S_i^- = \sqrt{\sum_{j=1}^{y}(v_{ij} - \lambda_j^-)^2}$$

6. **Calculate the relative closeness to the ideal solution:**

$$C_i = \frac{S_i^-}{S_i^+ + S_i^-}$$

7. **Rank the alternatives according to C_i in descending order.**

Step 2: Forming of the Weighted Normalized Decision Matrix
The table 6 contains the weighted normalized matrix.

Table 6
Normalized decision matrix with weight

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
λ_1	0.1495	0.1418	0.1279	0.1436
λ_2	0.1196	0.1476	0.1253	0.1265
λ_3	0.1112	0.1061	0.1201	0.1364
λ_4	0.1160	0.0968	0.1266	0.0853

Optimal best and optimal worst solutions
The optimal best and optimal worst solutions are presented in Table 7.

Table 7
Separation measure

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
A^+	0.1495	0.1476	0.1279	0.1436
A^-	0.1112	0.0968	0.1201	0.0853

The ultimate ranking options using TOPSIS is given in the Figure 3

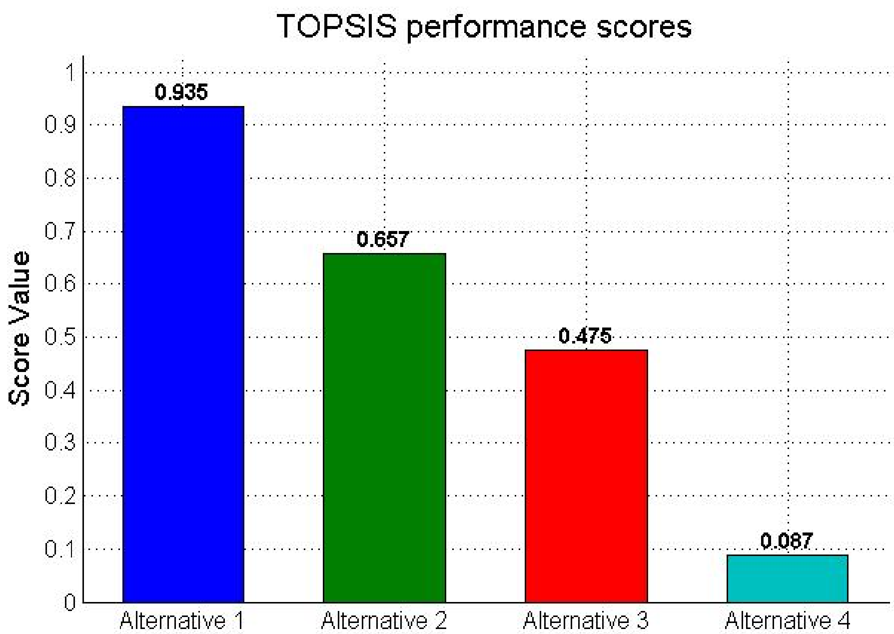

Fig. 3. Your descriptive caption goes here.

Algorithm 3 VIKOR Algorithm

Require: Decision matrix $D = [x_{ij}]$ of size $m \times n$, weights vector $w = [w_1, w_2, \ldots, w_n]$

Ensure: Ranking of alternatives

1: **Obtain crisp decision matrix:** Convert LDFZN based decision matrix into crisp matrix by applying score function on each LDFZN.

$$SF(\mathcal{Z}) = \frac{1}{2}\left[\left((\mu(\zeta\tau_j)\cdot\mu(\xi\tau_j)) - (\nu(\zeta\tau_j)\cdot\nu(\xi\tau_j))\right) + \left((\alpha(\zeta\tau_j)\cdot\alpha(\xi\tau_j) - (\gamma(\zeta\tau_j)\cdot\gamma(\xi\tau_j))\right)\right]$$

2: **Step 1: Determine the best and worst values for each criterion**

$$f_j^* = \max_i x_{ij}, \quad f_j^- = \min_i x_{ij}, \quad \text{for } j = 1, \ldots, n$$

3: **Step 2: Compute the utility measure S_i and the regret measure R_i**

$$S_i = \sum_{j=1}^{n} w_j \frac{f_j^* - x_{ij}}{f_j^* - f_j^-}$$

$$R_i = \max_j \left(w_j \frac{f_j^* - x_{ij}}{f_j^* - f_j^-} \right)$$

4: **Step 3: Compute the VIKOR index Q_i**

$$Q_i = v\frac{S_i - S^*}{S^- - S^*} + (1-v)\frac{R_i - R^*}{R^- - R^*}$$

where $S^* = \min_i S_i$, $S^- = \max_i S_i$, $R^* = \min_i R_i$, $R^- = \max_i R_i$, and v is the weight for the strategy of "the majority of criteria" (usually $v = 0.5$).

5: **Step 4: Rank the alternatives based on Q_i (primary), S_i, and R_i (secondary)**

6: **Step 5: Propose a compromise solution based on the following two conditions:**

- **Acceptable Advantage:** $Q(A^{(2)}) - Q(A^{(1)}) \geq DQ$, where $DQ = \frac{1}{m-1}$
- **Acceptable Stability:** $A^{(1)}$ must also be ranked the best by S or/and R.

VIKOR Method
The crisp decision matrix is same as of Table 4
Step 1: Ideal best and ideal worst.
The ideal best and ideal worst for each criterion is given in the Table 8

Table 8
Best (λ^+) and Worst (λ^-) values for each criterion

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
λ^+	0.6250	0.6400	0.4950	0.5050
λ^-	0.4650	0.4200	0.4650	0.3000

Step 2: Regret Measure R_i and Utility Measure S_i.
Regret measures and utility measures for each alternative are given in Table 9.

Table 9
Calculated regret (R_i) and utility (S_i) measures for alternatives

Alternative	S_i	R_i
λ_1	0.0284	0.0284
λ_2	0.3518	0.1953
λ_3	0.7350	0.2500
λ_4	0.7604	0.2500

Step 3: VIKOR index (Q_i).
The VIKOR index is given in the Table 10.

Table 10
VIKOR Index Values

Alternatives	VIKOR Index (Q_i)
λ_1	0.0000
λ_2	0.5975
λ_3	0.9827
λ_4	1.0000

Scores in Table 10 present inverted ranking. highest score presents lowest ranking and lowest score presents highest ranking. In order to visualize them we take reciprocal of scores and then divide them with those reciprocals with the sum of those reciprocals.

Given the VIKOR scores Q_i for each alternative λ_i, the ranking is determined through reciprocal normalization as follows:

Step 1: Compute Reciprocals

For each alternative λ_i, calculate the reciprocal of its Q_i score (with a small constant $\epsilon = 0.9$ for numerical stability):

$$r_i = \frac{1}{Q_i + \epsilon}, \quad \epsilon > 0 \tag{15}$$

Step 2: Sum of Reciprocals

Compute the total sum of all reciprocals:

$$S = \sum_{j=1}^{n} r_j = \sum_{j=1}^{n} \frac{1}{Q_j + \epsilon} \tag{16}$$

Step 3: Normalized Ranking Weights

Calculate the normalized weight for each alternative:

$$w_i = \frac{r_i}{S} = \frac{1/(Q_i + \epsilon)}{\sum_{j=1}^{n} 1/(Q_j + \epsilon)} \tag{17}$$

Step 4: Final Ranking

Rank alternatives in descending order of w_i, where higher w_i indicates better performance.
So the final ranking of alternatives is given in the figure 4

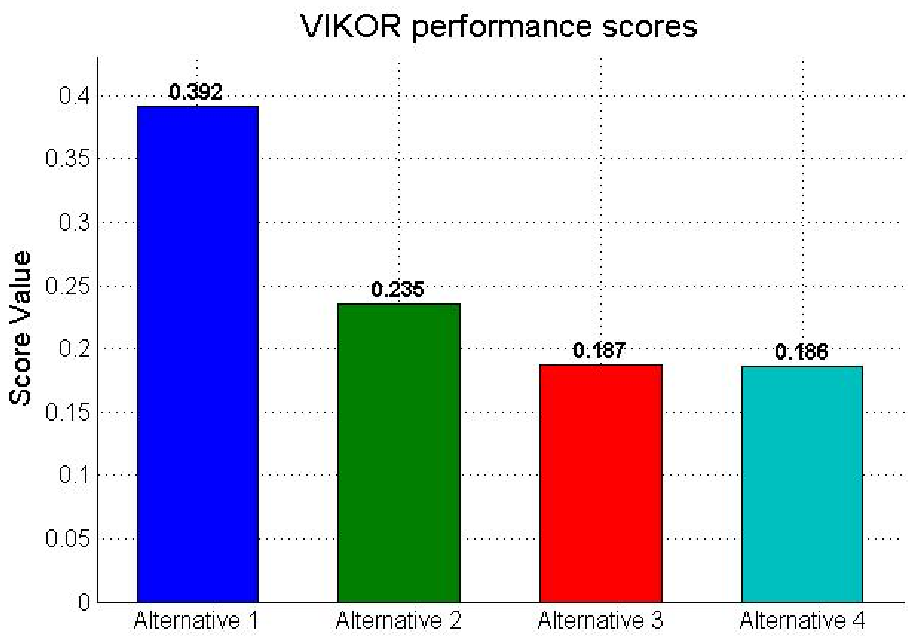

Fig. 4. Ranking from VIKOR.

8. sensitivity analysis

A comprehensive sensitivity analysis has been performed for LDFZNDWA operator. Ranking of alternatives is preserved for various values of Ω as exhibited in Figure 5. The ranking values are closer to 1 for higher values of Ω and tend to get close to each other as well.

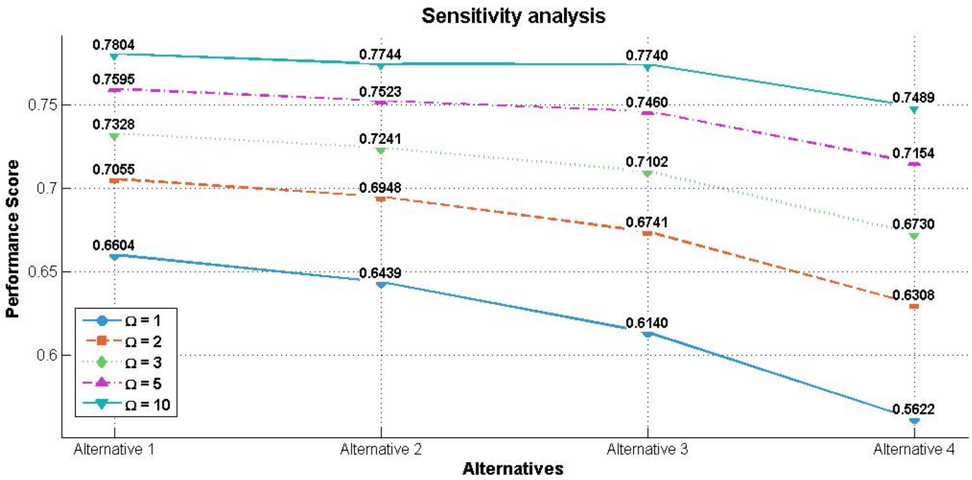

Fig. 5. Ranking from VIKOR.

9. Comparative analysis

A comprehensive comparative analysis has been performed with three distinct MCDM approaches, LDFZNDWA operator based decision making algorithm, TOPSIS and VIKOR algorithm in order to validate the effectiveness and versatility of proposed LDFZNDWA operator. The first approach independently applies the LDFZNDWA operator separately within the decision-making algorithm developed in this study. The second and third approaches uses LDFZNDWA operator with TOPSI and VIKOR respectively based on LDFZNs information. The results for the comparative analysis for these three approaches are shown in the Table 11 below.

Table 11
Comparison of proposed LDFZNDWA operator with VIKOR and TOPSIS

Method	Score				Ranking of Alternatives
	Score(λ_1)	Score(λ_2)	Score(λ_3)	Score(λ_4)	
LDFZNDWA operator	0.6604	0.6439	0.6140	0.5622	$\lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4$
TOPSIS	0.9353	0.6569	0.4752	0.0871	$\lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4$
VIKOR	0.3917	0.2354	0.1873	0.1856	$\lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4$

The final rankings of the three proposed algorithms are compared in Figure 6.

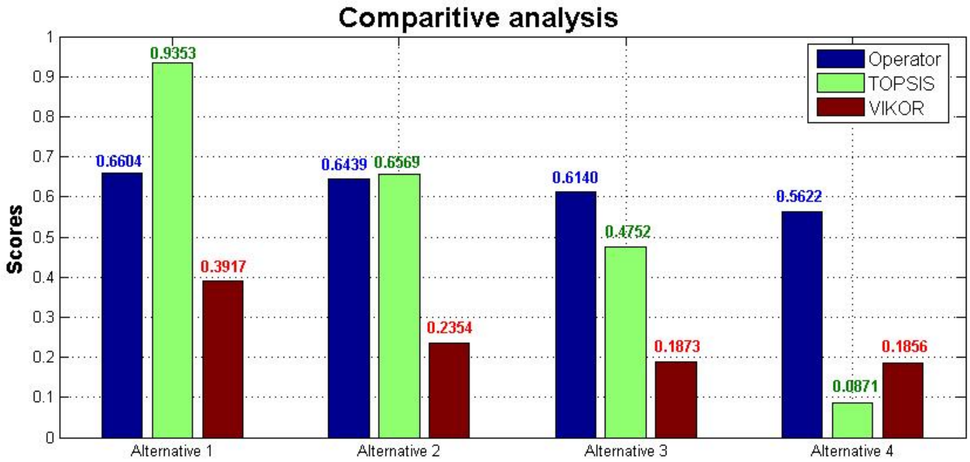

Fig. 6. Comparative analysis of proposed decision making algorithms

Compared to the existing MCDM models, our methodology handles uncertainty in a more enhanced way as we have used the LDFZNs which incorporate both membership and non membership along with their reliability values whereas the exiting models depend only on crisp or fuzzy values.

There is a very important point in there comparison that readers must keep in mind that topsis and vikor do not necessarily give similar ranking. In our supposed decision matrix all three algorithms are giving similar results but it is not always the case. We have conducted computations on many different supposed decision matrices and found that order of the rankings of alternative may vary in proposed decision making algorithms.

Different rankings may be produced by VIKOR and TOPSIS because they use different approaches in MCDM. Although both methods measure distances from ideal solutions, in TOPSIS's case, it only determines the distance between the options to the positive ideal solution without considering importance of each distance." On the other hand, VIKOR has a more refined approach by utilizing compromise solution which emphasizes increasing collective utility (average score) and minimizing individual regret (worst performance), hence it is applicable in conflict resolution situations. The essential disparity stems from VIKOR having a regret factor component providing compromise solutions resulting into differing rankings as opposed to the distance ranking used by TOPSIS; this difference is greater when dealing with closely performing alternatives or where decision makers focus on worst outcomes instead of average outcomes.

10. Managerial Implications

Following are the managerial applications of our study.

10.1 Enhanced Decision-Making Under Uncertainty

The LDFZNs framework will alters the landscape and will be an indispensable tool for managers confronting difficult decision making cases where the data are hazy and experts' opinions are spread out all over the place. In contrast to the traditional fuzzy techniques, the LDFZNs successfully express the degree of uncertainty and the level of confidence on such assessments. This makes them also highly relevant for important decision-making in areas such as urban development, improvement of infrastructure, and risk management. Take, for instance, municipal planners in Pakistan, who could analyze the costs and benefits of different transportation options—e.g., metro rail vs. bus rapid transit—taking into account budget constraints, environmental implications, and the realities of implementation, leading to better and more evidence-based policy.

10.2 *Improved Aggregation of Expert Judgments*

The LDFZNDWA operator proposes a systematic procedure to integrate the perspectives of several experts, even though the ideas that the experts suggest might be conflicting. It is particularly useful when there are many interested parties, such as in corporate strategic planning sessions or in talks regarding the public policy where one has to find a way of reconciling differing views. Managers can increase the viability of projects and the support of stakeholders by minimizing prejudice, which will result into judgments that approximate consensus by weighting expert assessments to the superiority of their predictions.

10.3 *Better Ranking of Alternatives Using Hybrid MCDM*

TOPSIS and VIKOR are used together with LDFZNs to offer an in-depth way of ranking selections in case of rival standards. As an example, economical factors such as cost, scalability, and environmental effects can be better evaluated when businesses select renewable energy projects. This systematic ranking process ensures that the solution that was chosen meets the short term demands as well as the long term sustainability goals since it does not involve subjective biases.

10.4 *Support for Sustainable and Cost-Effective Planning*

This approach strikes a compromise between environmental and economic considerations, which is crucial for decision-makers in developing nations like Pakistan. In order to manage risks and prioritize initiatives that yield the greatest advantages, decision-makers can measure uncertainty in cost projections, possible traffic reductions, and carbon emissions. In order to ensure that infrastructure investments fulfill international commitments like the Paris Agreement and the UN Sustainable Development Goals (SDG 11.2), this strategy is essential.

10.5 *Scalability Across Industries*

As research is only based on urban transportation, the presented LDFZN framework has broader applications in other sectors, including healthcare, finance, and supply chain management. Hospitals, for example, might use it to contemplate treatment options when the outcomes are uncertain. Investment professionals could use it to measure investment risk in turbulent markets. The flexibility of this approach as a general tool for complex decision making in the presence of uncertainty suggests that it can be a useful resource across all domains.

11. Conclusion

Capturing and managing complex uncertainty in MCDM was a challenge that this work successfully solved with LDFZNs. These numbers combine linear Diophantine structures with Z-number notions to present a novel method that goes beyond conventional fuzzy sets. The combination is helpful in providing a concurrent description of ambiguity and reliability in unreliable information. Significantly, the generalized operator LDFZNDWA has been considered. It was more than clear based on our mathematical reasoning that this operator fulfills some of the more important properties or characteristics; this includes but is by no way limited to, monotonicity, boundedness, idempotency, and closure. This proves that it is apposite and rational to merge LDFZN data. As part of our initiative in order to assess the viability of the suggested approach, we performed an intensive case study to select the best options in producing energy in Pakistan. We demonstrated how this framework can handle competing criteria and inherent uncertainties in a practical setting by combining LDFZNs and the LDFZNWA operator with a number of well-known MCDM methods, such as VIKOR and TOPSIS. By all means, this research provides a reliable and statistically sound instrument to the decision-makers. The LDFZN frame and LDFZNWA operator allow representing complex uncertainties more accurately in a complex situation where there is a multiplicity of interests and incomplete information. The outcome of this is making more responsible and sound decisions. Future research topics might include other LDFZN aggregation operators and knowledge of how the framework can be applied to other challenging decision-making areas.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

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