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Research on the Redefined Square Root Interval-valued Normal Pythagorean Fuzzy Multi-Attribute Decision-Making Model Based on Aggregation Operators

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ABSTRACT

In this paper, we construct new multiple attribute decision-making (MADM) problems using the redefined square root interval-valued normal Pythagorean fuzzy set (RSIVNPFS). The interval-valued Pythagorean fuzzy sets (IVPFSs) and square root PFSs are extended by the square RSIVNPFS. We introduce RSIVNPF weighted averaging (RSIVNPFWA), RSIVNPF weighted geometric (RSIVNPFWG), generalized RSIVNPFWA (RSGIVNPFWA), and generalized RSIVNPFWG (RSGIVNPFWG). Idempotence, boundedness, commutativity, and monotonicity in algebraic operations are all satisfied by RSIVNPFSs. We develop an algorithm for dealing with MADM problems using the aggregation operators (AOs). The applications of the Euclidean distance (ED) and the Hamming distance (HD) are described using examples from everyday scenarios. We also compare several suggested and current models to show the validity and applicability of the models. Our objective is to compare expert opinions with the criteria in order to determine the best option and to demonstrate the superiority and validity of the suggested AOs.

1. Introduction

Several theories have been developed to resolve uncertainty, including Zadeh's fuzzy set (FS) [1] theory, Atanassov's intuitionistic FS (IFS) [2] theory, interval-valued FS (IVFS) [3], vague set [4], and IVPyFS [5]. FSs are systems in which every element in the universe has a degree of belonging ranging from zero to one, and these grades are called membership grades (MG). Finally, Atanassov presented the idea of IFS logic, which states that the sum of MG and non-membership grades (NMG) cannot ex-

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ceed one. When membership degrees (MD) and non-membership degrees (NMD) are larger than one, making decisions becomes difficult. Yager created PFS [6] logic, which is distinguished by the condition that the square sum of MD and NMD must be less than or equal to one. Akram et al. [7] state that PFS has a variety of uses. Peng et al. [5] presented a Pythagorean fuzzy AO based on interval values. A few techniques for MADM based on the interval-valued Pythagorean fuzzy Einstein AO were presented by Rahman et al. [8]. Yang et al. [9] consolidated the MADM processes using IVPFS. Al-Shami et al. [10] studied square root FS (SFS) and weighted AOs. Ejegwa [11] proposed distance measures for IFSs, including Hamming distance (HD), Euclidean distance (ED), normalized HD, and normalized ED, to deal with MCDM and MADM challenges. Palanikumar et al. [12] used MADM to explain the extended Pythagorean neutrosophic normal set.

Zhou et al. [13] introduced the trust function-based divergence measurement of PFSs. Any two objects may be compared using a similarity measure to determine how similar they are to one another. Similarity assessments can be applied in a wide range of contexts. Oztaysi et al. [14] discussed the concept of social open innovation platform design for scientific curricula using the Pythagorean fuzzy analytic hierarchy technique. Song et al. [15] presented the idea of evaluating loan risk using the Pythagorean fuzzy analytic hierarchy method. Liu [16] discussed the concept of multiple q-rung orthopair fuzzy AOs and their use in MADM. A range of difficult situations arise in real-world scenarios, and data measures are useful for managing uncertain data. A bipolar FS using TOPSIS was proposed by Akram et al. in 2018 [17]. The MCGDM was introduced by Adeel et al. [18], along with the m-polar fuzzy linguistic TOPSIS. Practical applications of complex PFS were covered in Ullah et al. [19]. Section 1 contains the introduction. Section 2 provides an informative overview of PFS and its fundamental functions. Section 3 provides a description of RSNSNIVN. The generalization of the AO for RSNSNIVNs is covered in Section 4. Section 5 describes MADM based on RSNSNIVNs, and an algorithm and numerical example are presented. Section 6 provides the final conclusion. As a result, the research is likely to benefit the following areas.

1. Determine the distances using RSNSNIVS for the Euclidean (ED) and Hamming (HD) distances.

2. Four methods have been developed: RSIVNPF weighted averaging (RSIVNPFWA), RSIVNPF weighted geometric (RSIVNPFWG), generalized averaging (RSGIVNPFWA), and generalized geometric (RSGIVNPFWG).

3. An AO-based MADM technique is investigated with RSNSNIVS.

4. A numerical example is used to compare the suggested and current approaches.

5. The ideal values for RSIVNPFWA, RSIVNPFWG, RSGIVNPFWA, and RSGIVNPFWG are identified.

2. Preliminaries

This section provides a quick review of some of the fundamental terms used in our subsequent investigations. Let Ω be a universe set.

Definition 2.1 *The PFS* $\propto= \left\{\wp, \langle A^{\Im}_{\propto}(\wp), A^{\Re}_{\propto}(\wp)\rangle | \wp \in \Omega\right\}$, $A^{\Im}_{\propto}, A^{\Re}_{\propto} : \Omega \to [0,1]$ *denotes MD and NMD of* $\wp \in \Omega$ *to* $\propto$*, respectively, and* $0 \leq (A^{\Im}_{\propto}(\wp))^2 + (A^{\Re}_{\propto}(\wp))^2 \leq 1$. *For* $\propto= \langle A^{\Im}_{\propto}, A^{\Re}_{\propto}\rangle$ *is called a Pythagorean fuzzy number (PFN).*

Definition 2.2 *The SFS* $\propto= \left\{\wp, \langle A^{\Im}_{\propto}(\wp), A^{\Re}_{\propto}(\wp)\rangle | \wp \in \Omega\right\}$, $A^{\Im}_{\propto}, A^{\Re}_{\propto} : \Omega \to [0,1]$ *denotes MD and NMD of* $\wp \in \Omega$ *to* $\propto$*, respectively and* $0 \leq (A^{\Im}_{\propto}(\wp))^2 + \sqrt{A^{\Re}_{\propto}(\wp)} \leq 1$. *For* $\propto= \langle A^{\Im}_{\propto}, A^{\Re}_{\propto}\rangle$ *is called a square root fuzzy number (SFN).*

Definition 2.3 *[5] The PIVFS $\propto = \left\{\wp, \left\langle A^{\Im}_{\propto}(\wp), A^{\Re}_{\propto}(\wp)\right\rangle \middle| \wp \in \Omega\right\}$, where $A^{T}_{\propto}, A^{F}_{\propto} : \Omega \to Int([0,1])$ denotes MD and NMD of $\wp \in \Omega$ to $\propto$, respectively, and $0 \leq (A^{\Im u}_{\propto}(\wp))^2 + (A^{\Re u}_{\propto}(\wp))^2 \leq 1$. For $\propto = \left\langle \left[A^{\Im l}_{\propto}, A^{\Im u}_{\propto}\right], \left[A^{\Re l}_{\propto}, A^{\Re u}_{\propto}\right]\right\rangle$ is called a Pythagorean interval-valued fuzzy number (PIVFN).*

Definition 2.4 *[5] Let $\propto = \left\langle [A^{\Im l}, A^{\Im u}], [A^{\Re l}, A^{\Re u}]\right\rangle$, $\propto_1 = \left\langle [A^{\Im l}_1, A^{\Im u}_1], [A^{\Re l}_1, A^{\Re u}_1]\right\rangle$ and $\propto_2 = \left\langle [A^{\Im l}_2, A^{\Im u}_2], [A^{\Re l}_2, A^{\Re u}_2]\right\rangle$ be any three PIVFNs, and $\aleph \succ 0$. Then,*

1. $$\propto_1 \Box \propto_2 = \begin{bmatrix} \left[\sqrt{(A^{\Im l}_1)^2 + (A^{\Im l}_2)^2 - (A^{\Im l}_1)^2 \cdot (A^{\Im l}_2)^2}, \sqrt{(A^{\Im u}_1)^2 + (A^{\Im u}_2)^2 - (A^{\Im u}_1)^2 \cdot (A^{\Im u}_2)^2}\right], \\ \left[A^{\Re l}_1 \cdot A^{\Re l}_2, \ A^{\Re u}_1 \cdot A^{\Re u}_2\right] \end{bmatrix},$$
2. $$\propto_1 \Diamond \propto_2 = \begin{bmatrix} \left[A^{\Im l}_1 \cdot A^{\Im l}_2, \ A^{\Im u}_1 \cdot A^{\Im u}_2\right], \\ \left[\sqrt{(A^{\Re l}_1)^2 + (A^{\Re l}_2)^2 - (A^{\Re l}_1)^2 \cdot (A^{\Re l}_2)^2}, \sqrt{(A^{\Re u}_1)^2 + (A^{\Re u}_2)^2 - (A^{\Re u}_1)^2 \cdot (A^{\Re u}_2)^2}\right] \end{bmatrix},$$
3. $$\aleph \cdot \propto = \left[\left[\sqrt{1 - \left(1 - (A^{\Im l})^2\right)^{\aleph}}, \sqrt{1 - \left(1 - (A^{\Im u})^2\right)^{\aleph}}\right], \left[(A^{\Re l})^{\aleph}, (A^{\Re u})^{\aleph}\right]\right],$$
4. $$\propto^{\aleph} = \left[\left[(A^{\Im l})^{\aleph}, (A^{\Im u})^{\aleph}\right], \left[\sqrt{1 - \left(1 - (A^{\Re l})^2\right)^{\aleph}}, \sqrt{1 - \left(1 - (A^{\Re u})^2\right)^{\aleph}}\right]\right].$$

Definition 2.5 *For any SIVFN $\propto = \left\langle [A^{\Im l}, A^{\Im u}], [A^{\Re l}, A^{\Re u}]\right\rangle$, the score function of $\propto$ is*

$$S(\propto) = \frac{1}{2}\left((A^{\Im l})^2 + (A^{\Im u})^2 - \sqrt{A^{\Re l}} - \sqrt{A^{\Re u}}\right), \ S(\propto) \in [-1, 1],$$

Accuracy function of $\propto$ is

$$H(\propto) = \frac{1}{2}\left((A^{\Im l})^2 + (A^{\Im u})^2 + \sqrt{A^{\Re l}} + \sqrt{A^{\Re u}}\right), \ H(\propto) \in [0, 1].$$

3. Basic operations for RSIVNPFN

Definition 3.1 *The RSPIVS $\propto = \left\{\wp, \left\langle A^{\Im}_{\propto}(\wp), A^{\Re}_{\propto}(\wp)\right\rangle \middle| \wp \in \Omega\right\}$, where $A^{T}_{\propto}, A^{F}_{\propto} : \Omega \to Int([0,1])$ is called a TD and FD of $\wp \in \Omega$, respectively and $0 \leq (\daleth A^{\Im}_{\propto}(\wp))^2 + \sqrt{\daleth A^{\Re}_{\propto}(\wp)} \leq 1$ and $0 \leq (\daleth A^{\Im u}_{\propto}(\wp))^2 + \sqrt{\daleth A^{\Re u}_{\propto}(\wp)} \leq 1$, where $\daleth = \prod \left(A^{\Im}_{\propto}, A^{\Re}_{\propto}\right)$. For $\propto = \left\langle \left[A^{\Im l}_{\propto}, A^{\Im u}_{\propto}\right], \left[A^{\Re l}_{\propto}, A^{\Re u}_{\propto}\right]\right\rangle$ is called a RSPIVN.*

Definition 3.2 *For any RSPIVN $\propto = \left\langle \left[A^{\Im l}_{\propto}, A^{\Im u}_{\propto}\right], \left[A^{\Re l}_{\propto}, A^{\Re u}_{\propto}\right]\right\rangle$, the score function of $\propto$ is $S(\propto) = \frac{\chi}{2}\left(\frac{X}{2} + 1 - \frac{Z}{2}\right)$, where $X = (\daleth A^{\Im l})^2 + (\daleth A^{\Im u})^2$, $Z = \sqrt{\daleth A^{\Re l}} + \sqrt{\daleth A^{\Re u}}$ and $S(\propto) \in [-1, 1]$, where $\daleth = \prod \left(A^{\Im}_{\propto}, A^{\Re}_{\propto}\right)$.*

Definition 3.3 *Let $\propto = \left\langle (\chi, \eta); [A^{\Im l}, A^{\Im u}], [A^{\Re l}, A^{\Re u}]\right\rangle$, $\propto_1 = \left\langle (\chi_1, \eta_1); [A^{\Im l}_1, A^{\Im u}_1]\right.$ and $\propto_2 = \left\langle (\chi_2, \eta_2); [A^{\Im l}_2, A^{\Im u}_2], [A^{\Re l}_2, A^{\Re u}_2]\right\rangle$ be the any three RSIVNPFNs, and $\aleph \succ 0$ and $\daleth = \prod \left(A^{\Im}_{\propto}, A^{\Re}_{\propto}\right)$. Then,*

1. $\propto_1 \square \propto_2 = \left[\begin{array}{c} (\chi_1+\chi_2, \eta_1+\eta_2; \\ \left[\begin{array}{c} \left(\sqrt[2\aleph]{\daleth_i A_1^{\Im l}} + \sqrt[2\aleph]{\daleth_i A_2^{\Im l}} - \sqrt[2\aleph]{\daleth_i A_1^{\Im l}} \cdot \sqrt[2\aleph]{\daleth_i A_2^{\Im l}}\right)^{2\aleph}, \\ \left(\sqrt[2\aleph]{\daleth_i A_1^{\Im u}} + \sqrt[2\aleph]{\daleth_i A_2^{\Im u}} - \sqrt[2\aleph]{\daleth_i A_1^{\Im u}} \cdot \sqrt[2\aleph]{\daleth_i A_2^{\Im u}}\right)^{2\aleph} \end{array}\right], \\ \left[\daleth_i A_1^{\Re l} \cdot \daleth_i A_2^{\Re l}, \daleth_i A_1^{\Re u} \cdot \daleth_i A_2^{\Re u}\right] \end{array}\right],$

2. $\propto_1 \lozenge \propto_2 = \left[\begin{array}{c} (\chi_1 \cdot \chi_2, \eta_1 \cdot \eta_2; \left[\daleth_i A_1^{\Im l} \cdot \daleth_i A_2^{\Im l}, \daleth_i A_1^{\Im u} \cdot \daleth_i A_2^{\Im u}\right], \\ \left[\begin{array}{c} \left(\sqrt[2\aleph]{\daleth_i A_1^{\Re l}} + \sqrt[2\aleph]{\daleth_i A_2^{\Re l}} - \sqrt[2\aleph]{\daleth_i A_1^{\Re l}} \cdot \sqrt[2\aleph]{\daleth_i A_2^{\Re l}}\right)^{2\aleph}, \\ \left(\sqrt[2\aleph]{\daleth_i A_1^{\Re u}} + \sqrt[2\aleph]{\daleth_i A_2^{\Re u}} - \sqrt[2\aleph]{\daleth_i A_1^{\Re u}} \cdot \sqrt[2\aleph]{\daleth_i A_2^{\Re u}}\right)^{2\aleph} \end{array}\right] \end{array}\right],$

3. $\aleph \cdot \propto = \left[\begin{array}{c} (\aleph \cdot \chi, \aleph \cdot \eta); \\ \left[\left(1-\left(1-\sqrt[2\aleph]{\daleth_i A^{\Im l}}\right)^{\aleph}\right)^{2\aleph}, \left(1-\left(1-\sqrt[2\aleph]{\daleth_i A^{\Im u}}\right)^{\aleph}\right)^{2\aleph}\right], \\ \left[(\daleth_i A^{\Re l})^{\aleph}, (\daleth_i A^{\Re u})^{\aleph}\right] \end{array}\right],$

4. $\propto^{\aleph} = \left[\begin{array}{c} (\chi^{\aleph}, \eta^{\aleph}); \left[(\daleth_i A^{\Im l})^{\aleph}, (\daleth_i A^{\Im u})^{\aleph}\right], \\ \left[\left(1-\left(1-\sqrt[2\aleph]{\daleth_i A^{\Re l}}\right)^{\aleph}\right)^{2\aleph}, \left(1-\left(1-\sqrt[2\aleph]{\daleth_i A^{\Re u}}\right)^{\aleph}\right)^{2\aleph}\right] \end{array}\right].$

Definition 3.4 *For any two RSIVNPFNs* $\propto_1 = \left\langle (\chi_1, \eta_1; [A_1^{\Im l}, A_1^{\Im u}], [A_1^{\Re l}, A_1^{\Re u}] \right\rangle$ *and* $\propto_2 = \left\langle (\chi_2, \eta_2; [A_2^{\Im l}, A_2^{\Im u}], [A_2^{\Re l}, A_2^{\Re u}] \right\rangle$*. Then*

$$\nabla_E\Big(\propto_1, \propto_2\Big) = \frac{1}{2}\sqrt{\begin{array}{c} \left[\frac{1+\Upsilon_1-\beth_1}{2}\chi_1 - \frac{1+\Upsilon_2-\beth_2}{2}\chi_2\right]^2 \\ +\frac{1}{2}\left[\frac{1+\Upsilon_1-\beth_1}{2}\eta_1 - \frac{1+\Upsilon_2-\beth_2}{2}\eta_2\right]^2 \end{array}}$$

and

$$\nabla_H\Big(\propto_1, \propto_2\Big) = \frac{1}{2}\left[\begin{array}{c} \left|\frac{1+\Upsilon_1-\beth_1}{2}\chi_1 - \frac{1+\Upsilon_2-\beth_2}{2}\chi_2\right| \\ +\frac{1}{2}\left|\frac{1+\Upsilon_1-\beth_1}{2}\eta_1 - \frac{1+\Upsilon_2-\beth_2}{2}\eta_2\right| \end{array}\right]$$

where $\daleth = \prod\left(A_{\propto}^{\Im}, A_{\propto}^{\Re}\right)$ *and* $\Upsilon_1 = (\daleth_i A_1^{\Im l})^2 + (\daleth_i A_1^{\Im u})^2$, $\beth_1 = \sqrt{\daleth_i A_1^{\Re l}} + \sqrt{\daleth_i A_1^{\Re u}}$, $\Upsilon_2 = (\daleth_i A_2^{\Im l})^2 + (\daleth_i A_2^{\Im u})^2$, $\beth_2 = \sqrt{\daleth_i A_2^{\Re l}} + \sqrt{\daleth_i A_2^{\Re u}}$. *Since* $\nabla_E\Big(\propto_1, \propto_2\Big)$ *and* $\nabla_H\Big(\propto_1, \propto_2\Big)$ *are called ED and HD between* $\propto_1$ *and* $\propto_2$*, respectively.*

4. AOs via RSIVNPFN

We provide the new operators for RSIVNPFWA, RSIVNPFWG, RSGIVNPFWA, and RSGIVNPFWG.

4.1 RSIVNPF weighted averaging(RSIVNPFWA) operator

Definition 4.1 *Let* $\propto_i$ *be the set of RSIVNPFNs,* $W = (\varsigma_1, \varsigma_2, ..., \varsigma_n)$ *be the weight of* $\propto_i$*,* $\varsigma_i \geq 0$ *and* $\square_{i\mapsto 1}^{n}\varsigma_i = 1$*. Then RSIVNPFWA* $(\propto_1, \propto_2, ..., \propto_n) = \square_{i\mapsto 1}^{n}\varsigma_i\propto_i$*.*

Theorem 4.1 *Let $\propto_i$ be the set of RSIVNPFNs. Then* $RSIVNPFWA(\propto_1, \propto_2, ..., \propto_n)$=

$$\left[\begin{array}{c}\Big(\boxdot_{i\mapsto 1}^{n}\varsigma_i\chi_i, \boxdot_{i\mapsto 1}^{n}\varsigma_i\eta_i\Big);\\ \left[\left(1-\circledast_{i\mapsto 1}^{n}\Big(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im l})}\Big)^{\varsigma_i}\right)^{2\aleph}, \left(1-\circledast_{i\mapsto 1}^{n}\Big(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im u})}\Big)^{\varsigma_i}\right)^{2\aleph}\right],\\ \left[\circledast_{i\mapsto 1}^{n}(\daleth_i A_i^{\Re l})^{\varsigma_i}, \circledast_{i\mapsto 1}^{n}(\daleth_i A_i^{\Re u})^{\varsigma_i}\right]\end{array}\right].$$

Proof. The proof follows by induction method.
If $n = 2$, then RSIVNPFWA$(\propto_1, \propto_2) = \varsigma_1\propto_1\boxdot\varsigma_2\propto_2$, where

$$\varsigma_1\propto_1 = \left[\begin{array}{c}\Big(\varsigma_1\chi_1, \varsigma_1\eta_1\Big);\\ \left[\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_1^{\Im l})}\Big)^{\varsigma_1}\right)^{2\aleph}, \left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_1^{\Im u})}\Big)^{\varsigma_1}\right)^{2\aleph}\right],\\ \left[(\daleth_i A_1^{\Re l})^{\varsigma_1}, (\daleth_i A_1^{\Re u})^{\varsigma_1}\right]\end{array}\right]$$

$$\varsigma_2\propto_2 = \left[\begin{array}{c}\Big(\varsigma_2\chi_2, \varsigma_2\eta_2\Big);\\ \left[\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_2^{\Im l})}\Big)^{\varsigma_2}\right)^{2\aleph}, \left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_2^{\Im u})}\Big)^{\varsigma_2}\right)^{2\aleph}\right],\\ \left[(\daleth_i A_2^{\Re l})^{\varsigma_2}, (\daleth_i A_2^{\Re u})^{\varsigma_2}\right]\end{array}\right].$$

Now,

$$\varsigma_1\propto_1\boxdot\varsigma_2\propto_2 \;=\; \left[\begin{array}{c}\Big(\varsigma_1\chi_1+\varsigma_2\chi_2, \varsigma_1\eta_1+\varsigma_2\eta_2\Big);\\ \left[\begin{array}{c}\left(\begin{array}{c}\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_1^{\Im l})}\Big)^{\varsigma_1}\right)+\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_2^{\Im l})}\Big)^{\varsigma_2}\right)\\ -\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_1^{\Im l})}\Big)^{\varsigma_1}\right)\cdot\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_2^{\Im l})}\Big)^{\varsigma_2}\right)\end{array}\right)^{2\aleph},\\ \left(\begin{array}{c}\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_1^{\Im u})}\Big)^{\varsigma_1}\right)+\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_2^{\Im u})}\Big)^{\varsigma_2}\right)\\ -\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_1^{\Im u})}\Big)^{\varsigma_1}\right)\cdot\left(1-\Big(1-\sqrt[2\aleph]{(\daleth_i A_2^{\Im u})}\Big)^{\varsigma_2}\right)\end{array}\right)^{2\aleph}\end{array}\right],\\ \left[(\daleth_i A_1^{\Re l})^{\varsigma_1}(\daleth_i A_2^{\Re l})^{\varsigma_2}, (\daleth_i A_1^{\Re u})^{\varsigma_1}(\daleth_i A_2^{\Re u})^{\varsigma_2}\right]\end{array}\right]$$

$$\left[\begin{array}{c}\big(\varsigma_1\chi_1+\varsigma_2\chi_2,\ \varsigma_1\eta_1+\varsigma_2\eta_2\big);\\ \left[\begin{array}{c}\left(\big|-\sqrt[2\aleph]{(\daleth_i A_1^{\Im l})}\big|^{\varsigma_1}\cdot\big|-\sqrt[2\aleph]{(\daleth_i A_2^{\Im l})}\big|^{\varsigma_2}\right)^{2\aleph},\\ \left(\big|-\sqrt[2\aleph]{(\daleth_i A_1^{\Im u})}\big|^{\varsigma_1}\cdot\big|-\sqrt[2\aleph]{(\daleth_i A_2^{\Im u})}\big|^{\varsigma_2}\right)^{2\aleph}\end{array}\right],\\ \left[(\daleth_i A_1^{\Re l})^{\varsigma_1}\cdot(\daleth_i A_2^{\Re l})^{\varsigma_2},\ (\daleth_i A_1^{\Re u})^{\varsigma_1}\cdot(\daleth_i A_2^{\Re u})^{\varsigma_2}\right]\end{array}\right]$$

$$RSIVNPFWA(\propto_1, \propto_2) = \left[\begin{array}{c}\big(\boxdot_{i\mapsto 1}^{2}\varsigma_i\chi_i,\ \boxdot_{i\mapsto 1}^{2}\varsigma_i\eta_i\big);\\ \left[\left(1-\circledast_{i\mapsto 1}^{2}\Big(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im l})}\Big)^{\varsigma_i}\right)^{2\aleph},\ \left(1-\circledast_{i\mapsto 1}^{2}\Big(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im u})}\Big)^{\varsigma_i}\right)^{2\aleph}\right],\\ \left[\circledast_{i\mapsto 1}^{2}(\daleth_i A_i^{\Re l})^{\varsigma_i},\ \circledast_{i\mapsto 1}^{2}(\daleth_i A_i^{\Re u})^{\varsigma_i}\right]\end{array}\right]$$

Also $n \geq 3$, and hence
$RSIVNPFWA(\propto_1, \propto_2, \ldots, \propto_m) =$

$$\left[\begin{array}{c} \left(\square_{i\mapsto 1}^{m} \varsigma_i \chi_i,\ \square_{i\mapsto 1}^{m} \varsigma_i \eta_i\right); \\ \left[\left(1 - \circledast_{i\mapsto 1}^{m}\left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im l})}\right)^{\varsigma_i}\right)^{2\aleph},\ \left(1 - \circledast_{i\mapsto 1}^{m}\left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im u})}\right)^{\varsigma_i}\right)^{2\aleph}\right], \\ \left[\circledast_{i\mapsto 1}^{m} (\dashv_i A_i^{\Re l})^{\varsigma_i},\ \circledast_{i\mapsto 1}^{m} (\dashv_i A_i^{\Re u})^{\varsigma_i}\right] \end{array}\right]$$

If $n = m + 1$, then RSIVNPFWA $(\propto_1, \propto_2, ..., \propto_m, \propto_{m+1})$

$$= \left[\begin{array}{c} \left({}_{i\mapsto 1}^{m} \varsigma_i \chi_i + \varsigma_{m+1}\chi_{m+1}, {}_{i\mapsto 1}^{m} \varsigma_i \eta_i + \varsigma_{m+1}\eta_{m+1}\right); \\ \left[\left(\begin{array}{l} {}_{i\mapsto 1}^{m}\left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im l})}\right)^{\varsigma_i}\right) \\ + \left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_{m+1}^{\Im l})}\right)^{\varsigma_{m+1}}\right) \\ - \circledast_{i\mapsto 1}^{m}\left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im l})}\right)^{\varsigma_i}\right) \cdot \left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_{m+1}^{\Im l})}\right)^{\varsigma_{m+1}}\right) \end{array}\right)^{2\aleph}\right] \end{array}\right]$$

If $n = m + 1$, then RSIVNPFWA $(\propto_1, \propto_2, ..., \propto_m, \propto_{m+1})$

$$= \left[\begin{array}{c} \left(\square_{i\mapsto 1}^{m} \varsigma_i \chi_i + \varsigma_{m+1}\chi_{m+1}, \square_{i\mapsto 1}^{m} \varsigma_i \eta_i + \varsigma_{m+1}\eta_{m+1}\right); \\ \left[\left(\begin{array}{l} \square_{i\mapsto 1}^{m}\left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im u})}\right)^{\varsigma_i}\right) + \left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_{m+1}^{\Im u})}\right)^{\varsigma_{m+1}}\right) \\ - \circledast_{i\mapsto 1}^{m}\left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im u})}\right)^{\varsigma_i}\right) \cdot \left(1 - \left(1 - \sqrt[2\aleph]{(\dashv_i A_{m+1}^{\Im u})}\right)^{\varsigma_{m+1}}\right) \end{array}\right)^{2\aleph}\right], \\ \left[\circledast_{i\mapsto 1}^{m} (\dashv_i A_i^{\Re l})^{\varsigma_i} \cdot (\dashv_i A_{m+1}^{\Re l})^{\varsigma_{m+1}},\ \circledast_{i\mapsto 1}^{m} (\dashv_i A_i^{\Re u})^{\varsigma_i} \cdot (\dashv_i A_{m+1}^{\Re u})^{\varsigma_{m+1}}\right] \end{array}\right]$$

$$= \left[\begin{array}{c} \left(\square_{i\mapsto 1}^{m+1} \varsigma_i \chi_i, \square_{i\mapsto 1}^{m+1} \varsigma_i \eta_i\right); \\ \left[\left(1 - \circledast_{i\mapsto 1}^{m+1}\left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im l})}\right)^{\varsigma_i}\right)^{2\aleph},\ \left(1 - \circledast_{i\mapsto 1}^{m+1}\left(1 - \sqrt[2\aleph]{(\dashv_i A_i^{\Im u})}\right)^{\varsigma_i}\right)^{2\aleph}\right], \\ \left[\circledast_{i\mapsto 1}^{m+1} (\dashv_i A_i^{\Re l})^{\varsigma_i},\ \circledast_{i\mapsto 1}^{m+1} (\dashv_i A_i^{\Re u})^{\varsigma_i}\right] \end{array}\right].$$

Theorem 4.2 *If all $\propto_i$ are equal, then* RSIVNPFWA$(\propto_1, \propto_2, ..., \propto_n) = \propto$ *(idempotency property).*

Proof. Suppose that $(\chi_i, \eta_i) = (\chi, \eta)$, $[A_i^{\Im l}, A_i^{\Im u}] = [A^{\Im l}, A^{\Im u}]$ and $[A_i^{\Re l}, A_i^{\Re u}] = [A^{\Re l}, A^{\Re u}]$ and $\square_{i\mapsto 1}^{n} \varsigma_i = 1$.

Now,

$$RSIVNPFWA(\propto_1, \propto_2, ..., \propto_n)$$

$$= \left[\begin{array}{c} \left(\square_{i\mapsto 1}^{n}\varsigma_i\chi_i, \square_{i\mapsto 1}^{n}\varsigma_i\eta_i\right); \\ \left[\left(1-\circledast_{i\mapsto 1}^{n}\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im l})}\right)^{\varsigma_i}\right)^{2\aleph}, \left(1-\circledast_{i\mapsto 1}^{n}\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im u})}\right)^{\varsigma_i}\right)^{2\aleph}\right], \\ \left[\circledast_{i\mapsto 1}^{n}\left(\daleth_i A_i^{\Re l}\right)^{\varsigma_i}, \circledast_{i\mapsto 1}^{n}(\daleth_i A_i^{\Re u})^{\varsigma_i}\right] \end{array}\right],$$

$$= \left[\begin{array}{c} \left(\chi\square_{i\mapsto 1}^{n}\varsigma_i, \eta\square_{i\mapsto 1}^{n}\varsigma_i\right); \\ \left[\left(1-\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im l})}\right)^{\square_{i\mapsto 1}^{n}\varsigma_i}\right)^{2\aleph}, \left(1-\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im u})}\right)^{\square_{i\mapsto 1}^{n}\varsigma_i}\right)^{2\aleph}\right], \\ \left[(\daleth_i A_i^{\Re l})^{\square_{i\mapsto 1}^{n}\varsigma_i}, (\daleth_i A_i^{\Re u})^{\square_{i\mapsto 1}^{n}\varsigma_i}\right] \end{array}\right],$$

$$= \left[\begin{array}{c} (\chi, \eta); \\ \left[\left(1-\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im l})}\right)\right)^{2\aleph}, \left(1-\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im u})}\right)\right)^{2\aleph}\right], \\ \left[(\daleth_i A_i^{\Re l}), (\daleth_i A_i^{\Re u})\right] \end{array}\right].$$

$$= \propto.$$

Theorem 4.3 *Let $\propto_i, (j = 1, 2, ..., i_j)$ be a set of RSIVNPFWA, where*

$$\underbrace{\chi} = \inf \chi_{ij}, \quad \overbrace{\chi} = \sup \chi_{ij}, \quad \underbrace{\eta} = \sup \eta_{ij}, \quad \overbrace{\eta} = \inf \eta_{ij},$$

$$\underbrace{\daleth_i A^{\Im l}} = \inf \daleth_i A_{ij}^{\Im l}, \quad \overbrace{\daleth_i A^{\Im l}} = \sup \daleth_i A_{ij}^{\Im l},$$

$$\underbrace{\daleth_i A^{\Im u}} = \inf \daleth_i A_{ij}^{\Im u}, \quad \overbrace{\daleth_i A^{\Im u}} = \sup \daleth_i A_{ij}^{\Im u},$$

$$\underbrace{\daleth_i A^{\Re l}} = \inf \daleth_i A_{ij}^{\Re l}, \quad \overbrace{\daleth_i A^{\Re l}} = \sup \daleth_i A_{ij}^{\Re l},$$

$$\underbrace{\daleth_i A^{\Re u}} = \inf \daleth_i A_{ij}^{\Re u}, \quad \overbrace{\daleth_i A^{\Re u}} = \sup \daleth_i A_{ij}^{\Re u}.$$

Then, $\left\langle (\underbrace{\chi}, \underbrace{\eta}); [\underbrace{\daleth_i A^{\Im l}}, \underbrace{\daleth_i A^{\Im u}}], [\overbrace{\daleth_i A^{\Re l}}, \overbrace{\daleth_i A^{\Re u}}] \right\rangle \leq RSIVNPFWA(\propto_1, \propto_2, ..., \propto_n)$ $\leq \left\langle (\overbrace{\chi}, \overbrace{\eta}); [\overbrace{\daleth_i A^{\Im l}}, \overbrace{\daleth_i A^{\Im u}}], [\underbrace{\daleth_i A^{\Re l}}, \underbrace{\daleth_i A^{\Re u}}] \right\rangle$, *where* $1 \leq i \leq n,\ j = 1, 2, ..., i_j$ *(boundedness property).*

Proof. Since

$$\underbrace{\daleth_i A^{\Im l}} = \inf \daleth_i A_{ij}^{\Im l}, \quad \overbrace{\daleth_i A^{\Im l}} = \sup \daleth_i A_{ij}^{\Im l}, \quad \underbrace{\daleth_i A^{\Im u}} = \inf \daleth_i A_{ij}^{\Im u}, \quad \overbrace{\daleth_i A^{\Im u}} = \sup \daleth_i A_{ij}^{\Im u},$$

and

$$\underbrace{\Finv_i A^{\Im l}} \leq \Finv_i A_{ij}^{\Im l} \leq \overbrace{\Finv_i A^{\Im l}}, \quad \underbrace{\Finv_i A^{\Im u}} \leq \Finv_i A_{ij}^{\Im u} \leq \overbrace{\Finv_i A^{\Im u}}.$$

Now,

$$\begin{aligned}
\underbrace{\Finv_i A^{\Im l}} + \underbrace{\Finv_i A^{\Im u}} &= \left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\underbrace{\Finv_i A^{\Im l}})}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\underbrace{\Finv_i A^{\Im u}})}\right)^{\varsigma_i}\right)^{2\aleph} \\
&\leq \left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\Finv_i A_{ij}^{\Im l})}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\Finv_i A_{ij}^{\Im u})}\right)^{\varsigma_i}\right)^{2\aleph} \\
&\leq \left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\overbrace{\Finv_i A^{\Im l}})}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\overbrace{\Finv_i A^{\Im u}})}\right)^{\varsigma_i}\right)^{2\aleph} \\
&= \overbrace{\Finv_i A^{\Im l}} + \overbrace{\Finv_i A^{\Im u}}.
\end{aligned}$$

$$\begin{aligned}
\underbrace{\Finv_i A^{\Re l}} + \underbrace{\Finv_i A^{\Re u}} &= \circledast_{i=1}^{n}(\underbrace{\Finv_i A^{\Re l}})^{\varsigma_i} + \circledast_{i=1}^{n}(\underbrace{\Finv_i A^{\Re u}})^{\varsigma_i} \\
&\leq \circledast_{i=1}^{n}(\Finv_i A_{ij}^{\Re l})^{\varsigma_i} + \circledast_{i=1}^{n}(\Finv_i A_{ij}^{\Re u})^{\varsigma_i} \\
&\leq \circledast_{i=1}^{n}(\overbrace{\Finv_i A^{\Re l}})^{\varsigma_i} + \circledast_{i=1}^{n}(\overbrace{\Finv_i A^{\Re u}})^{\varsigma_i} \\
&= \overbrace{\Finv_i A^{\Re l}} + \overbrace{\Finv_i A^{\Re u}}.
\end{aligned}$$

Since

$$\underbrace{\chi} = \inf \chi_{ij}, \quad \overbrace{\chi} = \sup \chi_{ij}, \quad \underbrace{\eta} = \sup \eta_{ij}, \quad \overbrace{\eta} = \inf \eta_{ij},$$

and

$$\underbrace{\chi} \leq \chi_{ij} \leq \overbrace{\chi}, \quad \overbrace{\eta} \leq \eta_{ij} \leq \underbrace{\eta},$$

we have

$$\square_{i=1}^{n}\varsigma_i \underbrace{\chi} \leq \square_{i=1}^{n}\varsigma_i\chi_{ij} \leq \square_{i=1}^{n}\varsigma_i \overbrace{\chi}, \quad \square_{i=1}^{n}\varsigma_i \overbrace{\eta} \leq \square_{i=1}^{n}\varsigma_i\eta_{ij} \leq \square_{i=1}^{n}\varsigma_i \underbrace{\eta}.$$

Hence,

$$\begin{aligned}
&\frac{\square_{i=1}^{n}\varsigma_i \underbrace{\chi}}{2}\left[\begin{array}{c}\frac{\left(1-\circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\underbrace{\Finv_i A^{\Im l}})}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1-\circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\underbrace{\Finv_i A^{\Im u}})}\right)^{\varsigma_i}\right)^{2\aleph}}{2} \\ +1 - \frac{\sqrt{\circledast_{i=1}^{n}(\overbrace{\Finv_i A^{\Re l}})^{\varsigma_i}} + \sqrt{\circledast_{i=1}^{n}(\overbrace{\Finv_i A^{\Re u}})^{\varsigma_i}}}{2}\end{array}\right] \\
\leq\ &\frac{\square_{i=1}^{n}\varsigma_i\chi_{ij}}{2}\left[\begin{array}{c}\frac{\left(1-\circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\Finv_i A_{ij}^{\Im l})}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1-\circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\Finv_i A_{ij}^{\Im u})}\right)^{\varsigma_i}\right)^{2\aleph}}{2} \\ +1 - \frac{\sqrt{\circledast_{i=1}^{n}(\Finv_i A_{ij}^{\Re l})^{\varsigma_i}} + \sqrt{\circledast_{i=1}^{n}(\Finv_i A_{ij}^{\Re u})^{\varsigma_i}}}{2}\end{array}\right] \\
\leq\ &\frac{\square_{i=1}^{n}\varsigma_i \overbrace{\chi}}{2}\left[\begin{array}{c}\frac{\left(1-\circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\overbrace{\Finv_i A_{ij}^{\Im l}})}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1-\circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{(\overbrace{\Finv_i A_{ij}^{\Im u}})}\right)^{\varsigma_i}\right)^{2\aleph}}{2} \\ +1 - \frac{\sqrt{\circledast_{i=1}^{n}(\Finv_i A_{ij}^{\Re l})^{\varsigma_i}} + \sqrt{\circledast_{i=1}^{n}(\Finv_i A_{ij}^{\Re u})^{\varsigma_i}}}{2}\end{array}\right].
\end{aligned}$$

Therefore, $\left\langle (\underbrace{\chi}, \underbrace{\eta}); [\underbrace{A^{\Im l}}, \underbrace{A^{\Im u}}], [\overbrace{A^{\Re l}}, \overbrace{A^{\Re u}}] \right\rangle \leq RSIVNPFWA(\propto_1, \ldots, \propto_n)$
$\leq \left\langle (\overbrace{\chi}, \overbrace{\eta}); [\overbrace{A^{\Im l}}, \overbrace{A^{\Im u}}], [\underbrace{A^{\Re l}}, \underbrace{A^{\Re u}}] \right\rangle$.

Theorem 4.4 *(Monotonicity property) Let $\propto_i$ and W_i be families of RSIVNPFWAs. For any i, if $\chi_{t_{ij}} \leq \eta_{h_{ij}}$, $\sqrt{\daleth_i A^{\Im l}_{t_{ij}}} + \sqrt{\daleth_i A^{\Im u}_{t_{ij}}} \leq \sqrt{\daleth_i A^{\Im l}_{h_{ij}}} + \sqrt{\daleth_i A^{\Im u}_{h_{ij}}}$, $\daleth_i A^{\Re l}_{t_{ij}} + \daleth_i A^{\Re u}_{t_{ij}} \geq \daleth_i A^{\Re l}_{h_{ij}} + \daleth_i A^{\Re u}_{h_{ij}}$ or $\propto_i \leq W_i$, then $RSIVNPFWA(\propto_1, \ldots, \propto_n) \leq RSIVNPFWA(W_1, \ldots, W_n)$, where $j = 1, 2, \ldots, i_j$.*

Proof. For any i, $\chi_{t_{ij}} \leq \eta_{h_{ij}}$. Therefore,

$$\square_{i=1}^n \chi_{t_{ij}} \leq \square_{i=1}^n \eta_{h_{ij}}.$$

Also,

$$\sqrt{\daleth_i A^{\Im l}_{t_{ij}}} + \sqrt{\daleth_i A^{\Im u}_{t_{ij}}} \leq \sqrt{\daleth_i A^{\Im l}_{h_{ij}}} + \sqrt{\daleth_i A^{\Im u}_{h_{ij}}}.$$

Hence,

$$\begin{aligned}
&1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{t_i}} + 1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{t_i}} \\
&\geq 1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{h_i}} + 1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{h_i}}, \\
&\circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{t_i}}\right)^{\varsigma_i} + \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{t_i}}\right)^{\varsigma_i} \\
&\geq \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{h_i}}\right)^{\varsigma_i} + \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{h_i}}\right)^{\varsigma_i}, \\
&\left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{t_i}}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{t_i}}\right)^{\varsigma_i}\right)^{2\aleph} \\
&\leq \left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{h_i}}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{h_i}}\right)^{\varsigma_i}\right)^{2\aleph}.
\end{aligned}$$

For any i,

$$\daleth_i A^{\Re l}_{t_{ij}} + \daleth_i A^{\Re u}_{t_{ij}} \geq \daleth_i A^{\Re l}_{h_{ij}} + \daleth_i A^{\Re u}_{h_{ij}}.$$

Therefore,

$$1 - \frac{\circledast_{i=1}^n \daleth_i A^{\Re l}_{t_{ij}} + \circledast_{i=1}^n \daleth_i A^{\Re u}_{t_{ij}}}{2} \leq 1 - \frac{\circledast_{i=1}^n \daleth_i A^{\Re l}_{h_{ij}} + \circledast_{i=1}^n \daleth_i A^{\Re u}_{h_{ij}}}{2}.$$

Hence,

$$\begin{aligned}
&\frac{\square_{i=1}^n \chi tij}{2} \times \left[\frac{\left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{ti}}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{ti}}\right)^{\varsigma_i}\right)^{2\aleph}}{2} \atop + 1 - \frac{\sqrt{\circledast_{i=1}^n \daleth_i A^{\Re l}_{tij}} + \sqrt{\circledast_{i=1}^n \daleth_i A^{\Re u}_{tij}}}{2} \right] \\
&\leq \frac{\square_{i=1}^n \chi hij}{2} \times \left[\frac{\left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im l}_{hi}}\right)^{\varsigma_i}\right)^{2\aleph} + \left(1 - \circledast_{i=1}^n \left(1 - \sqrt[2\aleph]{\daleth_i A^{\Im u}_{hi}}\right)^{\varsigma_i}\right)^{2\aleph}}{2} \atop + 1 - \frac{\sqrt{\circledast_{i=1}^n \daleth_i A^{\Re l}_{hij}} + \sqrt{\circledast_{i=1}^n \daleth_i A^{\Re u}_{hij}}}{2} \right].
\end{aligned}$$

Hence,

$$RSIVNPFWA(\propto_1, \ldots, \propto_n) \leq RSIVNPFWA(W_1, \ldots, W_n).$$

4.2 RSIVNPF weighted geometric (RSIVNPFWG) operator

Definition 4.2 *Let $\propto_i$ be a set of RSIVNPFNs. Then the RSIVNPFWG operator is defined as*

$$RSIVNPFWG(\propto_1, \ldots, \propto_n) = \circledast_{i=1}^{n} \propto_i^{\varsigma_i}, \quad i = 1, \ldots, n.$$

Theorem 4.5 *Let $\propto_i$ be a set of RSIVNPFNs. Then*

$$\left[\begin{array}{c} \left(\circledast_{i=1}^{n} \chi_i^{\varsigma_i}, \circledast_{i=1}^{n} \eta_i^{\varsigma_i}\right); \left[\circledast_{i=1}^{n} (\daleth_i A_i^{\Im l})^{\varsigma_i}, \circledast_{i=1}^{n} (\daleth_i A_i^{\Im u})^{\varsigma_i}\right], \\ \left[\left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{\daleth_i A_i^{\Re l}}\right)^{\varsigma_i}\right)^{2\aleph}, \left(1 - \circledast_{i=1}^{n}\left(1 - \sqrt[2\aleph]{\daleth_i A_i^{\Re u}}\right)^{\varsigma_i}\right)^{2\aleph}\right] \end{array}\right].$$

Theorem 4.6 *If all $\propto_i$ are equal, then*

$$RSIVNPFWG(\propto_1, \propto_2, \ldots, \propto_n) = \propto.$$

Remark 4.1 *The RSIVNPFWG operator exhibits boundedness and monotonicity.*

4.3 Generalized RSIVNPFWA (RSGIVNPFWA) operator

Definition 4.3 *Let $\propto_i$ be a set of RSIVNPFNs. Then the RSGIVNPFWA operator is defined as*

$$RSGIVNPFWA(\propto_1, \ldots, \propto_n) = \left(\square_{i=1}^{n} \varsigma_i \propto_i^{\aleph}\right)^{1/\aleph}, \quad \aleph \neq 0.$$

Theorem 4.7 *Let $\propto_i$ be a set of RSIVNPFNs. Then*

$$RSGIVNPFWA(\propto_1, \ldots, \propto_n) =$$
$$\left[\begin{array}{c} \left((\square_{i=1}^{n} \varsigma_i \chi_i^{\aleph})^{1/\aleph}, (\square_{i=1}^{n} \varsigma_i \eta_i^{\aleph})^{1/\aleph}\right); \\ \left[(1 - \circledast_{i=1}^{n}(1 - \sqrt[2\aleph]{(\daleth_i A_i^{\Im l})^{\aleph}})^{\varsigma_i})^{2\aleph}, (1 - \circledast_{i=1}^{n}(1 - \sqrt[2\aleph]{(\daleth_i A_i^{\Im u})^{\aleph}})^{\varsigma_i})^{2\aleph}\right]; \\ \left[(1 - \sqrt[2\aleph]{\circledast_{i=1}^{n}((1 - \sqrt[2\aleph]{(\daleth_i A_i^{\Re l})})^{\aleph})^{2\aleph}})^{2\aleph}, (1 - \sqrt[2\aleph]{\circledast_{i=1}^{n}((1 - \sqrt[2\aleph]{(\daleth_i A_i^{\Re u})})^{\aleph})^{2\aleph}})^{2\aleph}\right] \end{array}\right].$$

Proof. First, observe that

$$\square_{i=1}^{n} \varsigma_i \propto_i^{\aleph} = \left[\begin{array}{c} \left(\square_{i=1}^{n} \varsigma_i \chi_i^{\aleph}, \square_{i=1}^{n} \varsigma_i \eta_i^{\aleph}\right); \\ \left[(1 - \circledast_{i=1}^{n} \sqrt[2\aleph]{(\daleth_i A_i^{\Im l})^{\aleph}})^{2\aleph}, (1 - \circledast_{i=1}^{n} \sqrt[2\aleph]{(\daleth_i A_i^{\Im u})^{\aleph}})^{2\aleph}\right]; \\ \left[\circledast_{i=1}^{n} ((1 - \sqrt[2\aleph]{(\daleth_i A_i^{\Re l})})^{\aleph})^{2\aleph}, \circledast_{i=1}^{n}((1 - \sqrt[2\aleph]{(\daleth_i A_i^{\Re u})})^{\aleph})^{2\aleph}\right] \end{array}\right].$$

Put $n = 2$ for simplicity.

$$\varsigma_1 \propto_1 \square \varsigma_2 \propto_2 = \left[\begin{array}{c} \left(\varsigma_1 \chi_1^{\aleph} + \varsigma_2 \chi_2^{\aleph},\ \varsigma_1 \eta_1^{\aleph} + \varsigma_2 \eta_2^{\aleph}\right) \\ \left[\begin{array}{c} \left(\sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_1^{\Im l})^{\aleph}})^{\varsigma_1}}^{2\aleph} + \sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_2^{\Im l})^{\aleph}})^{\varsigma_2}}^{2\aleph}\right. \\ \left. - \sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_1^{\Im l})^{\aleph}})^{\varsigma_1}}^{2\aleph} \cdot \sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_2^{\Im l})^{\aleph}})^{\varsigma_2}}^{2\aleph}\right)^{2\aleph}, \\ \left(\sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_1^{\Im u})^{\aleph}})^{\varsigma_1}}^{2\aleph} + \sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_2^{\Im u})^{\aleph}})^{\varsigma_2}}^{2\aleph}\right. \\ \left. - \sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_1^{\Im u})^{\aleph}})^{\varsigma_1}}^{2\aleph} \cdot \sqrt[2\aleph]{(\sqrt[2\aleph]{(\daleth_i A_2^{\Im u})^{\aleph}})^{\varsigma_2}}^{2\aleph}\right)^{2\aleph} \end{array}\right] \\ \left[\begin{array}{c} \left((\sqrt[2\aleph]{(\daleth_i A_1^{\Re l})})^{\aleph}\right)^{2\aleph\varsigma_1} \cdot \left((\sqrt[2\aleph]{(\daleth_i A_2^{\Re l})})^{\aleph}\right)^{2\aleph\varsigma_2}, \\ \left((\sqrt[2\aleph]{(\daleth_i A_1^{\Re u})})^{\aleph}\right)^{2\aleph\varsigma_1} \cdot \left((\sqrt[2\aleph]{(\daleth_i A_2^{\Re u})})^{\aleph}\right)^{2\aleph\varsigma_2} \end{array}\right] \end{array}\right]$$

In general,

$$\left[\begin{array}{c} \left(\square_{i\mapsto1}^{m}\varsigma_i\chi_i^{\aleph},\ \square_{i\mapsto1}^{m}\varsigma_i\eta_i^{\aleph}\right)\\ \left[\left(\circledast_{i\mapsto1}^{m}\left(\sqrt[2\aleph]{(\dashv_i A_i^{\Im l})^{\aleph}}\right)^{\varsigma_i}\right)^{2\aleph} \quad \left(\circledast_{i\mapsto1}^{m}\left(\sqrt[2\aleph]{(\dashv_i A_i^{\Im u})^{\aleph}}\right)^{\varsigma_i}\right)^{2\aleph}\right]\\ \left[\circledast_{i\mapsto1}^{m}\left(\left(\sqrt[2\aleph]{(\dashv_i A_i^{\Re l})}\right)^{\aleph}\right)^{2\aleph\varsigma_i} \quad \circledast_{i\mapsto1}^{m}\left(\left(\sqrt[2\aleph]{(\dashv_i A_i^{\Re u})}\right)^{\aleph}\right)^{2\aleph\varsigma_i}\right]\end{array}\right].$$

If $n = m + 1$, then

$$\square_{i\mapsto1}^{m}\varsigma_i\propto_i^{\aleph} + \varsigma_{m+1}\propto_{m+1}^{\aleph} = \square_{i\mapsto1}^{m+1}\varsigma_i\propto_i^{\aleph}.$$

Hence,

$$\square_{i\mapsto1}^{m+1}\varsigma_i\propto_i^{\aleph} = \left[\begin{array}{c} \left(\square_{i\mapsto1}^{m+1}\varsigma_i\chi_i^{\aleph},\ \square_{i\mapsto1}^{m+1}\varsigma_i\eta_i^{\aleph}\right)\\ \left[\left(1-\circledast_{i\mapsto1}^{m+1}\left(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Im l})^{\aleph}}\right)^{\varsigma_i}\right)^{2\aleph} \quad \left(1-\circledast_{i\mapsto1}^{m+1}\left(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Im u})^{\aleph}}\right)^{\varsigma_i}\right)^{2\aleph}\right]\\ \left[\circledast_{i\mapsto1}^{m+1}\left(\left(1-\left(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Re l})}\right)^{\aleph}\right)^{2\aleph}\right)^{\varsigma_i} \quad \circledast_{i\mapsto1}^{m+1}\left(\left(1-\left(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Re u})}\right)^{\aleph}\right)^{2\aleph}\right)^{\varsigma_i}\right]\end{array}\right]$$

Also,

$$\left(\square_{i\mapsto1}^{m+1}\varsigma_i\propto_i^{\aleph}\right)^{1/\aleph} =$$

$$\left[\begin{array}{c} \left(\square_{i\mapsto1}^{m+1}\varsigma_i\chi_i^{\aleph}\right)^{1/\aleph},\ \left(\square_{i\mapsto1}^{m+1}\varsigma_i\eta_i^{\aleph}\right)^{1/\aleph}\\ \left[\left(1-\circledast_{i\mapsto1}^{m+1}(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Im l})^{\aleph}})^{\varsigma_i}\right)^{2\aleph} \quad \left(1-\circledast_{i\mapsto1}^{m+1}(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Im u})^{\aleph}})^{\varsigma_i}\right)^{2\aleph}\right]\\ \left[\left(1-\sqrt[2\aleph]{\circledast_{i\mapsto1}^{m+1}\left(1-(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Re l})}))^{\aleph}\right)^{2\aleph\varsigma_i}}\right)^{2\aleph} \quad \left(1-\sqrt[2\aleph]{\circledast_{i\mapsto1}^{m+1}\left(1-(1-\sqrt[2\aleph]{(\dashv_i A_i^{\Re u})}))^{\aleph}\right)^{2\aleph\varsigma_i}}\right)^{2\aleph}\right]\end{array}\right]$$

Remark 4.2 *If $\aleph = 1$, then RSGIVNPFWA operator changes to the RSIVNPFWA operator.*

RSGIVNPFWA$(\propto_1, \propto_2, ..., \propto_n) = \propto$.

4.4 Generalized RSIVNPFWG (RSGIVNPFWG) operator

Definition 4.4 *Let $\propto_i$ be the set of RSIVNPFNs. Then*

$$\mathit{RSGIVNPFWG}(\propto_1, \propto_2, ..., \propto_n) = \frac{1}{\aleph}\left(\circledast_{i\mapsto1}^{n}(\aleph\propto_i)^{\varsigma_i}\right).$$

Theorem 4.8 *Let $\propto_i$ be the set of RSIVNPFNs. Then*

RSGIVNPFWG$(\propto_1, \propto_2, ..., \propto_n) =$

$$\left[\begin{array}{c} \left(\frac{1}{\aleph} \circledast_{i\mapsto 1}^{n} (\aleph\chi_i)^{\varsigma_i},\ \frac{1}{\aleph} \circledast_{i\mapsto 1}^{n} (\aleph\eta_i)^{\varsigma_i}\right) \\ \left[\left(1-\sqrt[2\aleph]{\circledast_{i\mapsto 1}^{n}\left(1-\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im l})}\right)^{\aleph}\right)^{2\aleph\varsigma_i}}\right)^{2\aleph} \quad \left(1-\sqrt[2\aleph]{\circledast_{i\mapsto 1}^{n}\left(1-\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Im u})}\right)^{\aleph}\right)^{2\aleph\varsigma_i}}\right)^{2\aleph}\right] \\ \left[\left(1-\circledast_{i\mapsto 1}^{n}\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Re l})^{\aleph}}\right)^{2\aleph\varsigma_i}\right)^{\aleph} \quad \left(1-\circledast_{i\mapsto 1}^{n}\left(1-\sqrt[2\aleph]{(\daleth_i A_i^{\Re u})^{\aleph}}\right)^{2\aleph\varsigma_i}\right)^{\aleph}\right] \end{array}\right]$$

Remark 4.3 *If $\aleph = 1$, then RSGIVNPFWG operator reduces to the RSIVNPFWG operator.*

5. RSIVNPFS concept via MADM

Let

$$\propto = \{\propto_1, \propto_2, ..., \propto_n\}, \quad C = \{C_1, C_2, ..., C_m\}, \quad w = \{\varsigma_1, \varsigma_2, ..., \varsigma_m\},$$

and let

$$\propto_{ij} = \left\langle (\chi_{ij}, \eta_{ij}); [A_{ij}^{\Im l}, A_{ij}^{\Im u}], [A_{ij}^{\Re l}, A_{ij}^{\Re u}] \right\rangle$$

be the RSIVNPFN of alternative $\propto_i$ in attribute C_j. Since

$$[A_{ij}^{\Im l}, A_{ij}^{\Im u}], [A_{ij}^{\Re l}, A_{ij}^{\Re u}] \in [0, 1]$$

and

$$0 \le (\daleth_i A_{ij}^{\Im u}(\wp))^2 + \sqrt{(\daleth_i A_{ij}^{Iu}(\wp))} + \sqrt{(\daleth_i A_{ij}^{\Re u}(\wp))} \le 1,$$

where

$$\daleth = \prod \left(A_{\propto}^{\Im}, A_{\propto}^{I}, A_{\propto}^{\Re}\right).$$

A decision is made using the algorithm described below.

5.1 *Algorithm*

Step-1: Form the RSIVNPF choice data.
Step-2: Normalize the decision matrix

$$\nabla = (\propto_{ij})_{n\times m} \quad \text{into} \quad \nabla = (\propto_{ij})_{n\times m}$$

by setting

$$\propto_{ij} = \left\langle (\chi_{ij}, \eta_{ij}); [\daleth_i A_{ij}^{\Im l}, \daleth_i A_{ij}^{\Im u}], [\daleth_i A_{ij}^{\Re l}, \daleth_i A_{ij}^{\Re u}] \right\rangle$$

with

$$\chi_{ij} = \frac{\chi_{ij}}{\sup_i(\chi_{ij})}, \quad \eta_{ij} = \frac{\eta_{ij}}{\sup_i(\eta_{ij})} \cdot \frac{\eta_{ij}}{\chi_{ij}}, \quad \daleth_i A_{ij}^{\Im l} = \daleth_i A_{ij}^{\Im l}, \quad \daleth_i A_{ij}^{\Im u} = \daleth_i A_{ij}^{\Im u}.$$

Step-3: Aggregate RSIVNPF values for attribute C_j in $\propto_i$:

$$\propto_i = \left\langle (\chi_i, \eta_i); [\daleth_i A_i^{\Im l}, \daleth_i A_i^{\Im u}], [\daleth_i A_i^{\Re l}, \daleth_i A_i^{\Re u}] \right\rangle.$$

Step-4: Determine the positive and negative ideal values:

$$\propto^+ = \left\langle (\sup_i \chi_{ij}, \inf_i \eta_{ij}); [1, 1], [0, 0] \right\rangle, \quad \propto^- = \left\langle (\inf_i \chi_{ij}, \sup_i \eta_{ij}); [0, 0], [1, 1] \right\rangle.$$

Step-5: Compute the Euclidean distance:

$$\nabla_i^+ = \nabla_E(\propto_i, \propto^+), \quad \nabla_i^- = \nabla_E(\propto_i, \propto^-).$$

Step-6: Compute the relative proximity:

$$\nabla_i^\star = \frac{\nabla_i^-}{\nabla_i^+ + \nabla_i^-}.$$

Step-7: Output the optimal result:

$$\sup_i \nabla_i^\star.$$

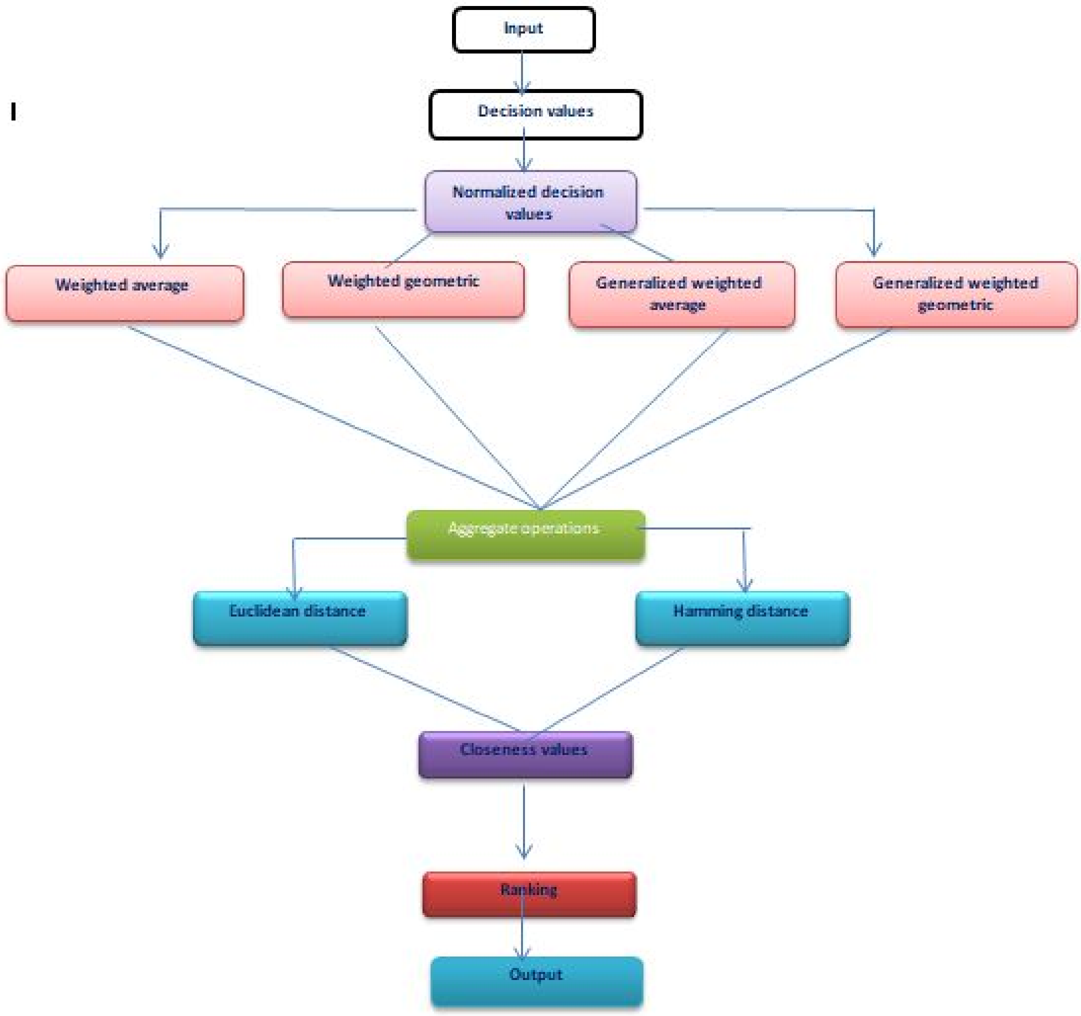

Fig. 1. Flowchart of the algorithm

5.2 *Real-life applications*

Pattern recognition is the act of detecting and categorizing patterns in data, both in human and computer settings. The three major kinds of artificial intelligence (supervised, unsupervised, and semi-supervised learning) each have their own way of finding patterns. Furthermore, pattern recognition algorithms may be classified based on their methodology, which includes template matching, neural network-based, statistical, and structural techniques. We have selected several pattern recognition tools at random: speech recognition, fingerprint recognition, handwriting recognition, template matching, and image processing. Four criterion categories are used: data collection and preparation, feature extraction, similarity evaluation, and decision-making, with respective weights $\{0.4, 0.3, 0.2, 0.1\}$. Based on expert assessments of the criteria, our goal is to select the best option. Tables 1 and 2 show the decision values.

Table 1
Decision values

	$\mathcal{C}_1$	$\mathcal{C}_2$
$\propto_1$	$\langle(0.8, 0.75); [0.44, 0.45], [0.45, 0.47]\rangle$	$\langle(0.7, 0.65); [0.41, 0.43], [0.43, 0.45]\rangle$
$\propto_2$	$\langle(0.75, 0.6); [0.45, 0.5], [0.3, 0.35]\rangle$	$\langle(0.8, 0.5); [0.5, 0.55], [0.25, 0.3]\rangle$
$\propto_3$	$\langle(0.7, 0.55); [0.4, 0.42], [0.25, 0.28]\rangle$	$\langle(0.9, 0.75); [0.4, 0.43], [0.38, 0.4]\rangle$
$\propto_4$	$\langle(0.75, 0.6); [0.35, 0.4], [0.5, 0.55]\rangle$	$\langle(0.75, 0.7); [0.5, 0.55], [0.45, 0.5]\rangle$
$\propto_5$	$\langle(0.8, 0.65); [0.4, 0.65], [0.3, 0.65]\rangle$	$\langle(0.75, 0.6); [0.55, 0.6], [0.25, 0.55]\rangle$

Table 2
Decision values

	$\mathcal{C}_3$	$\mathcal{C}_4$
$\propto_1$	$\langle(0.65, 0.6); [0.4, 0.41], [0.31, 0.32]\rangle$	$\langle(0.55, 0.5); [0.4, 0.41], [0.53, 0.55]\rangle$
$\propto_2$	$\langle(0.55, 0.5); [0.6, 0.64], [0.6, 0.64]\rangle$	$\langle(0.65, 0.45); [0.55, 0.58], [0.58, 0.6]\rangle$
$\propto_3$	$\langle(0.5, 0.3); [0.35, 0.38], [0.38, 0.4]\rangle$	$\langle(0.7, 0.35); [0.4, 0.41], [0.38, 0.4]\rangle$
$\propto_4$	$\langle(0.8, 0.65); [0.5, 0.55], [0.45, 0.5]\rangle$	$\langle(0.75, 0.65); [0.35, 0.38], [0.25, 0.3]\rangle$
$\propto_5$	$\langle(0.85, 0.7); [0.6, 0.65], [0.55, 0.65]\rangle$	$\langle(0.8, 0.45); [0.25, 0.28], [0.45, 0.55]\rangle$

Tables 3 and 4 provide a decision matrix for normalized data.

Table 3
Normalized decision values

	$\mathcal{C}_1$	$\mathcal{C}_2$
$\propto_1$	$\langle(1, 0.9375); [0.44, 0.45], [0.45, 0.47]\rangle$	$\langle(0.7778, 0.8048); [0.41, 0.43], [0.43, 0.45]\rangle$
$\propto_2$	$\langle(0.9375, 0.64); [0.45, 0.5], [0.3, 0.35]\rangle$	$\langle(0.8889, 0.4167); [0.5, 0.55], [0.25, 0.3]\rangle$
$\propto_3$	$\langle(0.875, 0.5762); [0.4, 0.42], [0.25, 0.28]\rangle$	$\langle(1, 0.8333); [0.4, 0.43], [0.38, 0.4]\rangle$
$\propto_4$	$\langle(0.9375, 0.64); [0.35, 0.4], [0.5, 0.55]\rangle$	$\langle(0.8333, 0.8711); [0.5, 0.55], [0.45, 0.5]\rangle$
$\propto_5$	$\langle(1, 0.7042); [0.4, 0.65], [0.3, 0.65]\rangle$	$\langle(0.8333, 0.64); [0.55, 0.6], [0.25, 0.55]\rangle$

Table 4
Normalized decision values

	$\mathcal{C}_3$	$\mathcal{C}_4$
$\propto_1$	$\langle(0.7647, 0.7912); [0.4, 0.41], [0.31, 0.32]\rangle$	$\langle(0.6875, 0.6993); [0.4, 0.41], [0.53, 0.55]\rangle$
$\propto_2$	$\langle(0.6471, 0.6494); [0.6, 0.64], [0.6, 0.64]\rangle$	$\langle(0.8125, 0.4793); [0.55, 0.58], [0.58, 0.6]\rangle$
$\propto_3$	$\langle(0.5882, 0.7571); [0.35, 0.38], [0.38, 0.4]\rangle$	$\langle(0.875, 0.2692); [0.4, 0.41], [0.38, 0.4]\rangle$
$\propto_4$	$\langle(0.9412, 0.7545); [0.5, 0.55], [0.45, 0.5]\rangle$	$\langle(0.9375, 0.8667); [0.35, 0.38], [0.25, 0.3]\rangle$
$\propto_5$	$\langle(1, 0.8235); [0.6, 0.65], [0.55, 0.65]\rangle$	$\langle(1, 0.3894); [0.25, 0.28], [0.45, 0.55]\rangle$

Table 5
RSIVNPFWG operator

Option	RSIVNPFWG operator ($\aleph = 1$)
$\overrightarrow{\propto_1}$	$\langle (0.855, 0.8446), [0.0121, 0.014], [0.0133, 0.0162] \rangle$
$\overrightarrow{\propto_2}$	$\langle (0.8523, 0.5588), [0.0374, 0.0562], [0.0091, 0.016] \rangle$
$\overrightarrow{\propto_3}$	$\langle (0.8551, 0.5588), [0.0087, 0.0116], [0.0044, 0.0062] \rangle$
$\overrightarrow{\propto_4}$	$\langle (0.907, 0.7549), [0.0129, 0.0216], [0.0112, 0.0204] \rangle$
$\overrightarrow{\propto_5}$	$\langle (0.95, 0.6773), [0.0153, 0.0421], [0.0062, 0.0777] \rangle$

Determine the optimal values for both positive and negative:

$$\overrightarrow{\propto}^{+} = \langle (0.95, 0.5588), 1, 1, 0 \rangle, \quad \overrightarrow{\propto}^{-} = \langle (0.8523, 0.8446), 0, 0, 1 \rangle.$$

The ED values for each choice under the various ideal values are as follows:

$$\nabla_1^{+} = 0.5314,\ \nabla_2^{+} = 0.5589,\ \nabla_3^{+} = 0.5585,\ \nabla_4^{+} = 0.5128,\ \nabla_5^{+} = 0.5009.$$

and

$$\nabla_1^{-} = 0.5214,\ \nabla_2^{-} = 0.4707,\ \nabla_3^{-} = 0.4711,\ \nabla_4^{-} = 0.5262,\ \nabla_5^{-} = 0.5309.$$

The following are the relative proximity values:

$$\nabla_1^{\star} = 0.4953,\ \nabla_2^{\star} = 0.4572,\ \nabla_3^{\star} = 0.4545,\ \nabla_4^{\star} = 0.5065,\ \nabla_5^{\star} = 0.5145.$$

The alternatives ranking is

$$\propto_5 \geq \propto_4 \geq \propto_1 \geq \propto_3 \geq \propto_2.$$

Digital image processing $\propto_5$ is the process of utilizing a computer to analyze and edit images in order to make them easier to read by humans and to extract visual information for activities such as rapid transmission, storage maintenance, and pictorial data extraction. Therefore, digital image processing $\propto_5$ is optimal.

5.3 *Comparison of proposed methods and existing models*

The RSIVNPFWA, RSIVNPFWG, RSGIVNPFWA, and RSGIVNPFWG techniques are utilized to represent the information above using ED and HD, respectively. The distances are classified in Tables 6 and 7.

Table 6
Proposed existing methods for values

$\aleph = 1$	WA	WG	GWA	GWG
TOPSIS–Euclidean distance (proposed)	$\propto_5 \geq \propto_4 \geq \propto_1$ $\propto_3 \geq \propto_2$	$\propto_5 \geq \propto_4 \geq \propto_1$ $\propto_2 \geq \propto_3$	$\propto_5 \geq \propto_4 \geq \propto_1$ $\propto_3 \geq \propto_2$	$\propto_5 \geq \propto_4 \geq \propto_1$ $\propto_2 \geq \propto_3$
TOPSIS–Hamming distance (proposed)	$\propto_1 \geq \propto_3 \geq \propto_4$ $\propto_2 \geq \propto_5$	$\propto_4 \geq \propto_3 \geq \propto_5$ $\propto_2 \geq \propto_1$	$\propto_1 \geq \propto_3 \geq \propto_4$ $\propto_2 \geq \propto_5$	$\propto_4 \geq \propto_3 \geq \propto_5$ $\propto_2 \geq \propto_1$

Table 7
Proposed existing methods for values

$\aleph = 1$	WA	WG	GWA	GWG
Euclidean distance [9]	$\propto_5 \geq \propto_4 \geq \propto_2$ $\propto_1 \geq \propto_3$	$\propto_5 \geq \propto_4 \geq \propto_2$ $\propto_1 \geq \propto_3$	$\propto_5 \geq \propto_4 \geq \propto_2$ $\propto_1 \geq \propto_3$	$\propto_5 \geq \propto_4 \geq \propto_2$ $\propto_1 \geq \propto_3$
Hamming distance [9]	$\propto_5 \geq \propto_4 \geq \propto_2$ $\propto_1 \geq \propto_3$	$\propto_5 \geq \propto_4 \geq \propto_1$ $\propto_2 \geq \propto_3$	$\propto_5 \geq \propto_4 \geq \propto_2$ $\propto_1 \geq \propto_3$	$\propto_5 \geq \propto_4 \geq \propto_1$ $\propto_2 \geq \propto_3$

Figure 2 shows the EDs of suggested and present models.

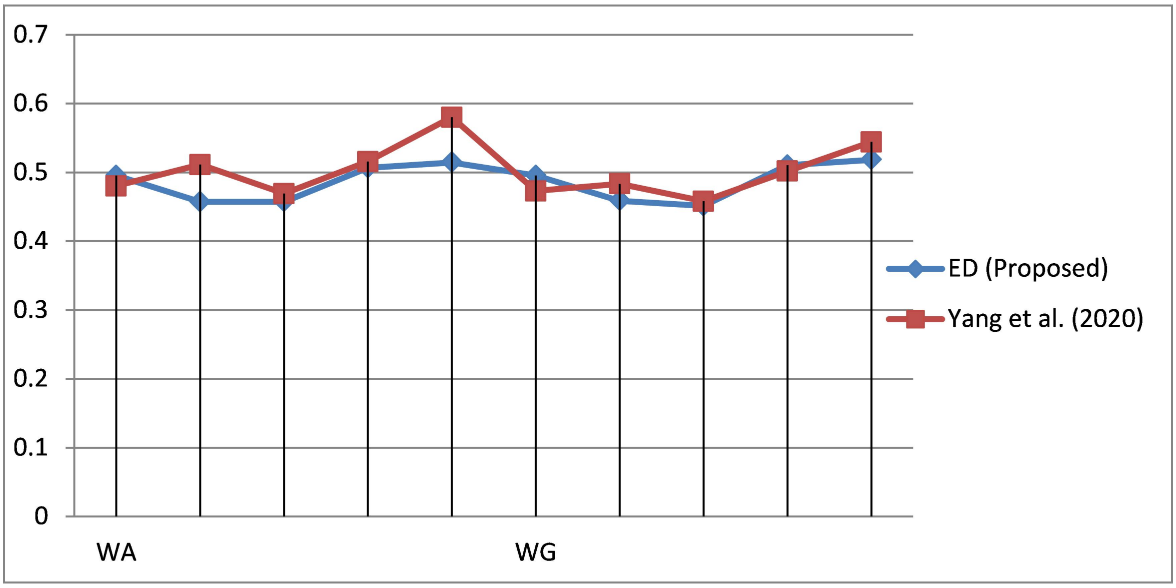

Fig. 2. Graphical representation of EDs

Figure 3 shows the HDs of suggested and present models.

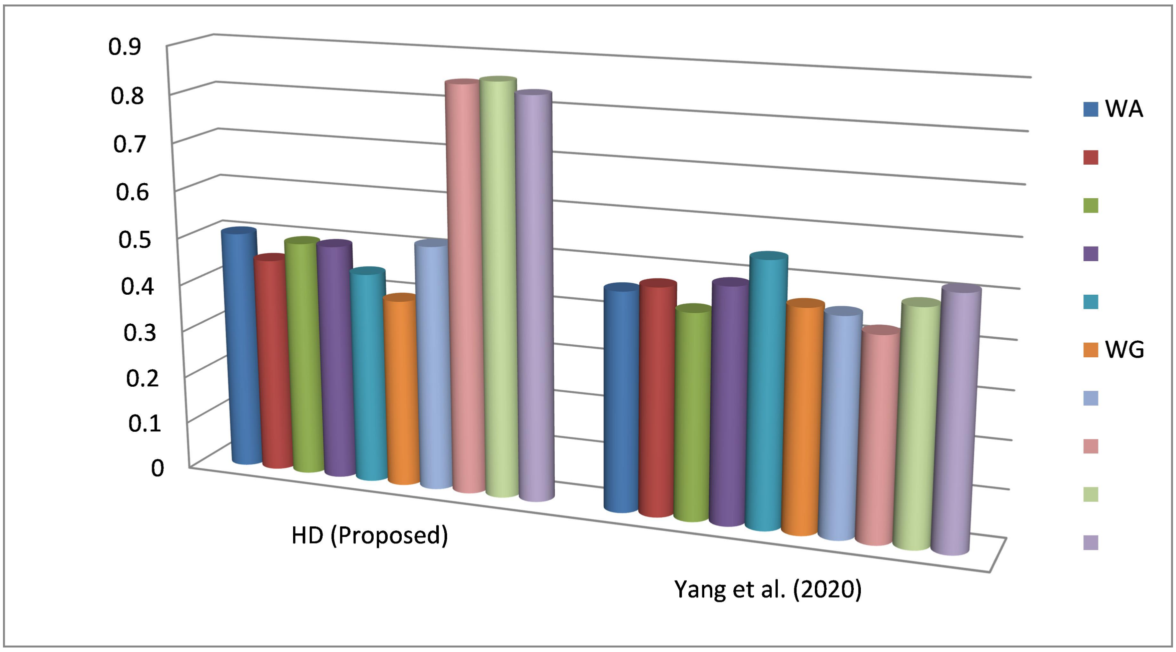

Fig. 3. Graphical representation of HDs

6. Conclusion

This research introduces new weighted operators, including averaging and geometric operators. These operators possess several properties, such as monotonicity, boundedness, idempotency, associativity, and commutativity. A variety of criteria for the aggregate operator have been considered. We explored several aggregation approaches for RSIVNPFSs and presented some findings in the process. We proposed AO principles for RSIVNPFWA, RSIVNPFWG, RSGIVNPFWA, and RSGIVNPFWG operators to be used in situations where confusing or conflicting information exists, giving individuals the option of making an appropriate response. Prioritizing the various possibilities is important before ranking them. The generalized values of $\aleph$ and their impact on the ranking of alternatives are also discussed. Before setting $\aleph$ values, decision makers should evaluate the real circumstances to determine the optimal option.

We plan to continue studying symmetric operators, power operators, Hamacher operators, Dombi operators, and Einstein operators, among other topics. The following subjects will be covered in further detail: (1) The Diophantine FS with interaction AOs and the Diophantine square root FS. (2) Diophantine normals defined in different sets, including the Diophantine normal spherical set and the Diophantine normal FS.

Conflicts of Interest

The authors declare no conflict of interest.

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